Corps et modèles—Essai sur l'histoire de l'algèbre réelle. By Hourya Sinaceur. Paris: Librairie Philosophique J. Vrin. 1991. 496 pp.

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Mathematical analysis in France in the first half of the 19th century; abstract algebra in Germany in the 1920s; the study of quadratic forms from the 1960s on; metamathematical research on the semantics of mathematical theories in the U.S.A. from the early 1930s to the 1970s. These four apparently distant mathematical subjects are the main topics discussed in Hourya Sinaceur's *Corps et Modèles*. This is in no sense a collection of essays on separate topics brought together into a single volume. Rather, this book presents the reader with an organic and intriguing story concerning the development of several important mathematical ideas, interconnected through one single central axis: Charles Sturm's theorem on the location of roots of a polynomial equation.

On May 25, 1829, a memoir by Charles F. Sturm was read before the Royal Academy of Sciences in Paris. It presented an easily performable algorithm that gave a precise answer to the following question: Given any polynomial V of degree m with real coefficients, and any two real numbers a and b, how many real roots of the equation $V = 0$ lie between a and b? The publication of Sturm's algorithm aroused great interest in wide mathematical circles. Between 1835 and 1853 Sturm himself, as well as other leading mathematicians such as Liouville, Sylvester, Cayley, Kronecker, and Hermite, published additional works that elaborated on Sturm's original method. Over the following decades and until about 1910, a steady flow of publications on this subject continued to appear.

During the first part of the 19th century, especially in France, mathematical analysis was the main focus of interest in the whole discipline of mathematics. It completely dominated all mathematical activity: it determined the standard methods of research, the styles of teaching and writing, and the important open problems to be addressed. The term "algebra" denoted a specific portion of this mainstream concern, involving a well-defined, classical task: the study of polynomial forms and of polynomial equations in all of its manifestations, emphasizing the study of all available methods for solving particular equations, either by radicals or by approximation. This was the conceptual framework with which Sturm's theorem was naturally associated at the time of its publication.

The first part of Sinaceur's book ("Le théorème d'algèbre de Ch. F. Sturm") discusses the development of this result from Sturm's early treatments, until Charles Hermite's publication of two memoirs on the theorem, in 1853. Sinaceur points out the reduced attention that the theorem has been given in modern historical accounts, and contrasts it with the enthusiasm accorded to it in the 19th century. She explains in detail Sturm's mathematical background and the close connections of Sturm's theorem with his other works on mathematical analysis (in particular on differential equations in connection with celestial and analytical mechanics). She also describes the significant influences on Sturm of mathematicians working on similar issues, like Laplace and Lagrange, and especially of Fourier with his work on the propagation of heat. Under the outlook shared by all these mathematicians it is natural to consider Sturm's theorem, and any eventual generalization of it, as one that should be proven by means of continuity arguments. It is precisely the dissolution of this apparently obvious connection that will characterize the work discussed in the second part of the book. Sinaceur shows here, however, that the theorem was gradually "algebraized" still in the context of analysis, in the hands of Sylvester and Hermite. These two mathematicians increasingly relied in their proofs on the specific properties satisfied by the polynomials as such, and avoided whenever possible the kind of considerations of continuity originally applied by Sturm.

The developments described in the second part of the book ("La construction algébrique des corps réels par Emil Artin et Otto Schreier") take place in Germany of the twenties, in the framework of the then newly consolidated structural approach to algebra. The central figures, Artin and Schreier, worked at that time at the University of Hamburg, but the kind of research they were conducting was typical of other centers in Germany, and especially so of Göttingen, among the mathematicians associated to the circle of Emmy Noether. During the 1920s the discipline of algebra underwent deep changes that completely transformed its scope, its methods and its aims: the research of algebraic structures came to the focus of its interest. The study of polynomial equations and their solvability—the main task of algebra during the 19th century—became subsidiary to the study of structures. In 1930 this new view of algebra was presented to a large mathematical audience in the form of a textbook: the now classic *Moderne Algebra* by Bartel L. van der Waerden. Van der Waerden's book was directly modeled on lectures by Artin and by Noether, which the author had attended in previous years in Hamburg and in Göttingen, respectively. In his book, van der Waerden addressed the issue of finding the roots of polynomial equations in only three short sections of a chapter on Galois theory. On the face of it, this context seems less natural for highlighting the significance of Sturm's theorem, at least in its original formulation. As Sinaceur shows in her book, however, the theorem played a central role in one of the main proofs of Artin and Schreier's theory of real fields.

A major step towards the consolidation of the structural approach in algebra had been the publication, in 1910, of Ernst Steinitz's abstract theory of fields. It provided a model that was later followed by many mathematicians—among them Emmy Noether in her own theory of ideals. Steinitz's work made clear the extent to which many important properties of fields, such as the field of real numbers, as well as many properties of polynomials, can be directly derived from algebraic properties without recourse to "analytical considerations." But Steinitz's work was far from providing a *complete* characterization of the real numbers in these terms. In fact, before the work of Artin and Schreier it was common belief that no such characterization would be possible.

Artin and Schreier's publications of 1926 and 1927 on "real fields" came both as a surprise and as a confirmation of the power of the new approach to algebra. Artin and Schreier showed that such a basic property of the field of real numbers as the relation of order defined on it—which was traditionally associated with topological considerations and with continuity arguments—could, in fact, be derived from basic "algebraic" concepts alone: "=," "0," "1," and the four arithmetical operations. Artin and Schreier defined a field as "real," whenever -1 cannot be expressed as a sum of squares of numbers in that field.

The second part of *Corps et modèles* describes the background to Artin and Schreier's theory and their own elaboration of it. It provides an account of the attempts to deal with the notion of order in the system of real numbers, from Dedekind through Hilbert and until the American postulationalists of the first decade of our century (especially Edward Huntington). It includes an account of Steinitz's theory of fields, and of the influence of Hilbert on the development of modern algebra. In particular, it stresses the importance of the 17th of Hilbert's 1900 list of 23 problems ("Can every definite quadratic form be expressed as a quotient of sums of squares of forms?") as a major drive leading to Artin and Schreier's definition of real fields. In fact, Sinaceur claims that this problem is a foremost example of a leading idea of Hilbert that has been strongly confirmed by recent mathematical research: the fecundity of problems that have to be addressed from a multidisciplinary perspective and using diverse mathematical techniques taken from separate domains of research. Sinaceur's book is by its very nature a case study of a development of this kind.

How are the developments described in the second part connected with Sturm's theorem? In the first place, the achievement of Artin and Schreier lies precisely in the fact that most of the important theorems on real functions and their roots that were known from classical analysis, could now be proven in the theory of (closed) real fields, based only on algebraic considerations. That is the case with Sturm's theorem, but also, e.g., with the theorems of Rolle and Bolzano. But there is a second important connection. Artin and Schreier defined the real closure of any field: a fundamental result of their theory is the existence and uniqueness (up to isomorphism) of this closure for any given field. Artin and Schreier's proof of uniqueness—as well as later proofs of van der Waerden and of Nathan Jacobson—relies on a certain version of Sturm's theorem. This centrality of the theorem for the theory subsequently led to the research of a further question: is the theorem also *necessary* for the proof of uniqueness? The interest in this question remained alive in certain mathematical quarters. One of them was the study of quadratic forms.

The third, shorter, part of Sinaceur's book is thus devoted to developments concerning the connections between the theory of real fields and the theory of quadratic forms after 1960. It concerns the creation of a new autonomous discipline: real algebraic geometry. Among others, it discusses two affirmative answers to the above-mentioned question of whether the uniqueness of the real closure can be proved without relying on Sturm's theorem.

In the fourth part ("Logique et algèbre réelle"), Sinaceur discusses the connection between logic and the theory of real fields. In the early 1930s, Alfred Tarski started his metamathematical investigations concerning the semantics of mathematical theories. Eventually, these works led to the creation of model theory as an autonomous branch of metamathematics. The evolution of the ideas developed by Tarski and his successors in this context are thus part of the history of 20th-century mathematical logic but, as it happens, they are at the same time closely connected with both Artin and Schreier's theory and with Sturm's theorem.

This part of the book comprises two sections. In the first one, the author describes Tarski's method of elimination of quantifiers. Sinaceur discusses here the origins of Tarski's metamathematics and its close relation to Sturm's theorem. The second section deals with the relation of algebra and logic, and with the role of the latter as a tool of discovery in mathematics. Special attention is given to Abraham Robinson's concept of model-completeness and his generalization of elimination procedures to domains other than closed real fields. These concepts also enabled a generalized construction of the kind stipulated by Hilbert's 17th problem.

Sinaceur claims that the developments of model theory since 1930 led to a fundamental change of relations between logic and mathematics in general, and particularly between the former and algebra. At variance with Hilbert, who explicitly insisted upon the need to separate mathematical from metamathematical reasoning, Tarski intended model theory to have meaningful, direct applications on ordinary mathematics. Tarski's was, in this sense, the first metamathematical work that was neither mathematical epistemology nor proof theory, but rather, according to Sinaceur's account, a "general theory of mathematical theories."

One aspect in which the book leaves the reader wanting concerns the contextualization of the mathematical developments covered in parts three and four. Whereas the French mathematical environment in which Sturm produced his work is widely discussed, concerning both institutions and ideas, and whereas the German tradition leading to Artin and Schreier's contribution is explained in great detail, one would like to know much more about parallel aspects of the work of Tarski and of Robinson. This kind of gap is perhaps unavoidable given the scope of the topics covered, but is nevertheless felt.

The developments described in *Corps et modèles* bear not just upon the accumulation of new mathematical concepts, theorems, and proofs over a relatively long period of time. Rather, they concern the way in which the internal organization of mathematics into subdisciplines and the interrelation among the latter change over time, and how these changes are themselves connected to the creation of new ideas and to the reevaluation of existing ones. By focusing on one such process of change—involving 18th-century analysis, early 20th-century algebra, and mid-20th-century logic—Sinaceur underscores with her book the historical insight that may be expected from paying closer attention to them.

The account presented in *Corps et modèles* is remarkable in the range and diversity of topics covered and in the unity of its conception. Any reader interested in the history of modern mathematics, but especially so in the history of algebra and of logic, will find in it a fascinating story and at the same time a profuse source of information and reference. The prospective reader is likely to be unacquainted with the technical details of the issues discussed in one or more parts of the book, and must therefore be ready to invest considerable effort in mastering them. Yet, Sinaceur's explanations are clear and thorough at both the technical and historical levels, and thus the reader's effort will be rewarded.