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Origins of the Structural Approach to Algebra

The structural approach to algebra was presented for the first time in its full-fledged conception in B.L. van der Waerden's *Moderne Algebra*, published in 1930. Under this approach, several mathematical constructs (e.g., groups, rings, fields, hypercomplex systems, etc.) are seen as diverse manifestations of a single, general underlying notion: the notion of an algebraic structure. The aim of algebra, under this conception, is to investigate these constructs from a unified point of view, asking similar questions about all of them, and trying to answer these questions with the help of the same kinds of tools in all cases.

Prior to the structural approach, algebra was conceived as a discipline dealing with the theory of polynomial forms and polynomial equations, including all available methods known to solve the latter. A thorough knowledge of the properties of the various systems of numbers (rational, real and complex) was assumed to provide the foundation for the elaboration of algebraic knowledge. Arithmetic was the foundation of algebra. Under the structural approach, the conceptual hierarchy is turned upside down, and all systems of numbers appear as no more than particular instances of algebraic structures or of combinations thereof: the rational numbers, for instance, are the ring of quotients of a rational domain, while the real numbers are an ordered, "real" field. Polynomials with real coefficients are studied as a certain kind of ring, whose properties derive from "structural" considerations. Certain properties of all these entities, which were formerly considered as a matter of course as part of algebraic research (e.g., continuity, density, order), are not of the concern of algebra anymore, and they are left to the study of other mathematical disciplines.

Van der Waerden wrote his book under direct influence of Emmy Noether and Emil Artin, whose courses he had recently attended in Göttingen and Hamburg, respectively. Noether and Artin consolidated in their separate work in the 1920s the elements with which van der Waerden reconstructed algebra in his book. However, most of the concepts that in the latter's presentation of algebra appear as different manifestations of the underlying, general idea of an algebraic structure developed separately over the second half of the nineteenth century, still within the pre-structural image of algebra that dominated that period of time. Thus, although the structural approach to algebra does not exist as such before the 1920s, its roots are clearly manifest since the mid-1860s. The slow and complex process, whereby the separate definition and study of individual concepts eventually became instances of algebraic strucures, is more clearly manifest in two main contexts: the evolution of Galois theory and the theory of algebraic number fields.

The publication of Galois's works in 1846, by Joseph Liouville, opened new perspectives for dealing with the theory of polynomial equations. However, his ideas were in many respects so innovative that it took considerable time and effort to assimilate them properly into the existing body of knowledge, and in fact, they were incorporated into algebra in different ways.

Richard Dedekind was among the first to attempt a systematic clarification of the theory, in a series of lectures taught at Göttingen in 1856-57. He followed closely both Galois's original results and his approach, stressing the parallel relationship between the Galois group and its subgroups, on the one hand, and the field of rationals and its successive extensions by addition of roots, on the other hand. As the theories of groups and of fields evolved gradually into full blown mathematical domains, the principles behind Dedekind's approach became the basis for the standardly accepted conception of Galois theory: following the work of Artin in the mid 1920s, the subject-matter of the theory became the interrelation between these two structures —in the spirit originally formulated by Dedekind— while the question of the solvability of equations by radicals turned into a particular application of it. Still, one can notice significant differences between Dedekind's original conception and that implied by Artin's work, especially concerning the different roles played by the various concepts involved.

In Dedekind's treatment of Galois theory, the main subject matter is the interrelations between subfields of the system of complex numbers ("rational domains"), not of abstract fields. Groups, on the contrary, appear as no more than a tool —albeit a very effective and innovative one— that can be used to solve important problems related to that system. Based on the work of Galois, Dedekind explained in a very clear, almost axiomatic (in the modern sense) way the nature of this tool and how its use is justified in this context. This kind of analysis is absent form his discussions of rational domains. Moreover, whenever he discussed rational domains, Dedekind focused on the *properites of their elements*, whereas concerning groups, what interested him were always the *groups as such.* Thus, in many respects, Dedekind's groups and fields are mathematical entities belonging to different conceptual levels.

But Dedekind's conception of Galois theory was quite idiosyncratic, and in fact it had no immediate influence on teaching and research. The conceptions typically reflected in textbooks of the period are even more distant from the idea of an algebraic structure than his. In his *Cours d'algèbre supérieure*, for instance, Joseph A. Serret did not change the conception of algebra as the discipline of polynomial equations and polynomial forms, even when, in the last of its three editions (1866), it became the first university textbook to publish an exposition of Galois theory. Serret's book presented this theory as an auxiliary tool for actually determining the roots of a given equation. He did not discuss the concept of group on its own, and he formulated the main theorems of Galois theory in the traditional language of solvability theory, going back to the works of Lagrange and Abel.

Somewhat different was the approach followed by Jordan in his *Traité des substitutions et des équations algébriques*, the first textbook that deliberately and explicitly connected the theory of equations and the theory of permutations. Jordan, for instance, introduced the important concept of composition series. He had originally defined it in an article published in 1869 (though not in terms of groups of permutations, but rather in terms of algebraic equations alone), and in the *Traité* the concept already appears as a purely group-theoretical one. However, lacking the concept of quotient group, Jordan focused on the quotients of the *orders* of two successive groups in a composition series. He proved that the number and values of these quotients (except perhaps for their order) are an invariant of the group. The task of providing the broader, more abstract setting for the theorem had to be postponed almost twenty years until Otto Hölder completed in 1899 the second part of the proof of the theorem named today after both mathematicians. The different treatments of the concept of group by Serret, Jordan and Hölder illustrate the kind of processes undergone in the last third of the past century by ideas that eventually came to be related to structural algebra.

Towards the end of the century Heinrich Weber published his *Lehrbuch der Algebra,* which soon became the standard text of the field. It included most of the recent advances in algebra missing from its predecessors, and yet it preserved the main traits of the classical nineteenth-century image of algebra as the science of polynomial equations and polynomial forms.

Weber had published in 1893 an important article containing an exposition of the theory of Galois in the most general terms known to that date, deeply influenced by Dedekind's point of view, and including a joint definition of groups and fields, both in completely abstract terms. In many respects, Weber's article represents the first truly modern published presentation of Galois theory. In particular, it introduced all the elements needed to establish the isomorphism between the group of permutations of the roots of the equation and the group of automorphisms of the splitting field that leave the elements of the base field invariant. This presentation of Weber suggests —in a natural way— the convenience of adopting an abstract formulation of the central concepts of group and field. Moreover, it stresses the importance of the interplay between structural properties of both entities. It therefore implied an important move towards the understanding of the idea of an algebraic structure, and towards the adoption of the structural approach.

The innovative approach adopted by Weber in this paper, however, had minimal direct influence on the algebraic research of other, contemporary mathematicians, and, more strikingly perhaps, it did not even influence the perspective adopted in Weber's own textbook, published soon thereafter. As late as 1924, a new textbook of algebra was published by Robert Fricke under the explicit influence of Weber's textbook, in which even the "new abstract approach" of Weber's article is totally absent.

The bulk of the first volume of Weber's *Lehrbuch* discusses the problem of finding the roots of polynomial equations, assuming a thorough knowledge of the properties of the different systems of numbers. The concept of root of an equation is discussed in terms that later became foreign to the structural approach to algebra:

limits, continuity, ε–δ arguments etc. Likewise, we find a discussion of the theorem of Sturm, concerning the quantity of real roots of a given polynomial equation that lie between two given real numbers. This theorem is related to the use of derivatives and other analytical tools, and so are interpolation techniques and Newton's approximation method, which Weber treated in detail.

Galois theory is introduced in Weber's book only after having discussed, in nearly five hundred pages, other techniques associated with the resolution of polynomial equations. Even then, the fields are mostly seen as sets of complex numbers closed under the four operations, and only groups of substitutions are defined in the first place. The focus of interest does not lie in the study of the properties of the group of permutations as such, but only insofar as it sheds light on the classical theory of equations. A definition of group in terms similar to those of Weber's article of 1893 appears only in the second volume of the *Lehrbuch*, followed by an elementary, though rather comprehensive, discussion of the results of the theory as known to that date.

In 1908 Weber published the third volume of the *Lehrbuch*, which was, in fact, a second edition of his book on elliptic functions and algebraic numbers, first published in 1891. Under the typical images of algebra of the turn of the century this domain found a natural place in a textbook of algebra. It thus provides a good example of how the eventual prevalence of the structural approach to algebra implied a redefinition of the borders of the domain: in *Moderne Algebra*, and in other textbooks modeled after it, one finds no treatment of elliptic functions and of their relations to problems of algebraic number theory.

By the end of the century, group theory was the paradigm of an abstractly developed theory. It was, perhaps, the only discipline in contemporary mathematical research that may be really qualified as "structural." Research on groups had increasingly focused on questions that we recognize today as structural, and, at the same time, the possibility of defining the concept abstractly had been increasingly acknowledged. More importantly, the idea that two isomorphic groups are in essence one and the same mathematical construct had been increasingly absorbed. Weber's 1893 article exemplifies clearly this situation. Yet, in Weber's *Lehrbuch*, group theory plays a role which, at most, may be described as ambiguous regarding the overall picture of algebra. For, although in the second volume, the theory of groups is indeed presented as a mathematical domain of intrinsic interest for research, and many techniques and problems are presented in an up-to-date, structurally-oriented fashion, the theory appears in the first volume, like for Dedekind, as no more than a tool of the theory of equations (albeit, it is now clear, a central one). Weber's textbook, and much more so his 1893 article, bring to the fore the interplay between groups and fields abstractly considered more than any former, similar work. However, in spite of this, the classical conceptual hierarchy that viewed algebra as based on the given properties of the number systems was not changed, or even questioned, in either of these two works.

Weber's *Lehrbuch* became the standard German textbook on algebra and underwent several reprints. Its influence can be easily detected through the widespread adoption of a large portion of the terminology introduced in it. Thus, the image of algebra conveyed by Weber's book was to dominate the algebraic scene for almost thirty years, until van der Waerden's presentation of the new, structural image of algebra. But obviously, influential as the latter was on the further development of algebra, it did not immediately obliterate Weber's influence, which can still be traced to around 1930 and perhaps even beyond.

Dedekind's work also provides the best perspective to examine the role of concepts such as fields, modules and ideals, in the framework of the theory of algebraic number fields. Working separately and following different approaches, both Dedekind and Kronecker attempted in their respective works after 1865 to develop a complete theory of unique factorization in algebraic number fields, elaborating on ideas initially introduced by Kummer. Roughly speaking, Kronecker's approach may be described as more algorithmic-oriented, whereas Dedekind's can be characterized as more conceptual. Dedekind introduced concepts such as ideals, fields, and modules, and strongly relied on their use in order to provide a solid basis for his theory. In this sense, his work does put forward what, retrospectively, can be seen as a marked structural spirit. However, a close examination of the way in which all these concepts appear in his work reveals that, like in the case of Galois theory, they are used with very different conceptions in mind.

Fields provide the basic conceptual framework for the study of algebraic integers in Dedekind's work, while modules and ideals are *tools* for investigating the properties of the factorization of those integers. Dedekind's modules and ideals are not "algebraic structures" similar to yet another "structure", fields. They are not "almost-fields", failing to satisfy one of the postulates that define the latter. While the numbers belonging to the fields remain themselves the focus of Dedekind's interest, the properties of the numbers that constitute the modules and ideals are never investigated; what interests in this case are the properties of the *collections of numbers*. Therefore, the study of modules and ideals depends on operations such as intersection, union and inclusion, and not on the operational relations among the numbers contained in them.

Ideals never appear in Dedekind's work as a special kind of substructure of the more general algebraic structure of a ring. He defined *Ordnungen*, which are formally equivalent to rings, but these *Ordnungen* do not provide a general framework for the study of ideals like rings do in the structural conception of algebra. An ideal is not a distinguished subdomain of an *Ordnung*. Likewise, when dealing with modules, Dedekind never mentioned the basic fact that —from an abstract perspective— they are in fact groups of numbers. Neither did he apply to modules results formerly obtained for groups.

. Dedekind also defined *Dualgruppen*, which are formally equivalent to lattices, yet differ from the latter in the way their essence and function are conceived. In defining *Dualgruppen*, Dedekind did not take an arbitrary set and endow it with an abstractly defined relation of order or with two abstractly defined operations, in order to check which theorems can be deduced for such a construct. Rather, he was motivated by the desire to improve the proofs of certain identities already known to hold for modules. He defined an "algebra of modules" and established abstract, logical interdependencies between specific properties of certain operations arising in it. He defined specific kinds of *Dualgruppen* (*Idealgruppe* and *Modulgruppe*) as collections whose inclusion properties, unions and intersections, satisfy certain identities (or "axioms"), and investigated under what conditions those identities are mutually equivalent. Finally, he studied the specific example of a certain mathematical entity, which had arisen in his previous work —the 28-element

Dualgruppe— and extended the previous analysis of the logical interdependence of the axioms to include now an additional one: the chain axiom. Although Dedekind did provide examples of how *Dualgruppen* appear in the algebra of logic, fields, etc., he did not establish any direct connection between them and the other "algebraic structures." Dedekind's *Dualgruppen* do not appear in is work as a further instance of one and the same species of mathematical entities like fields, groups, or ideals.

The true conceptual affinity of Dedekind's ideals is not to other "algebraic structures" appearing in his work, but rather to two concepts introduced by him in a separate context: cuts and chains. Cuts, chains, and ideals are three concepts aiming at providing conceptual foundations for the systems of real, natural, and algebraic numbers, respectively. Each of these concepts is meant to allow for the proof of some basic results, from which the most important facts concerning the domains in question may be derived. The concept of cut is meant to elucidate the idea of continuity in the system of real numbers. The concept of chain is meant to elucidate the idea of the sequence of natural numbers. In the same vein, the concept of ideal was conceived to elucidate the most important problem concerning the domain of algebraic integers, namely, the question of unique factorization. Moreover, Dedekind's treatment of these three concepts is similar in many ways, although they were published at different times.

Like with Galois theory, Dedekind's idiosyncratic work in this domain provides a limiting case and a litmus test for evaluating the degree of acceptance of "structural" or "abstract" ideas in late-nineteenth-century algebra. The prevalence in twentieth-century algebra of the point of view initially introduced by Dedekind is in complete opposition to its contemporary acceptance. In the case of algebraic number theory, the watershed is very clealry recognizable: the publication in 1897 of Hilbert's *Zahlbericht*. Although conceived as a summary of the results produced by Kummer, Dedekind and Kronecker in this domain, the *Zahlbericht* was not a survey in the usual sense of the term. Hilbert did produce an impressive and exhaustive systematization of the existing results of the discipline, but he also added many important results of his own. It became the standard reference text for mathematicians working in algebraic number theory, and since Hilbert basically adopted Dedekind's approach as the leading one, its publication turned out to be a decisive factor for the consequent dominance of Dedekind's perspective over that of Kronecker within the discipline.

The approach underlying the *Zahlbericht* is similar to Dedekind's also concerning the interrelation among the basic concepts and tools, and the same attitude is manifest in Hilbert's article on the theory of fields of algebraic numbers, published in 1900 in the *Encyclopädie der Mathematischen Wissenschaften*. He defined rings, using this term for the first time, but he did so in a purely number-theoretic context: a ring in Hilbert's sense is a system of algebraic integers of the given field, closed under the three mentioned operations. Hilbert defined an ideal of a ring as any system of algebraic integers belonging to the ring, such that any linear combination of them (with coefficients in the ring) belongs itself to the ideal. Hilbert quoted in this context several results from Dedekind's theory of ideals, and he never described a ring as a group endowed with a second operation, or as a field whose division fails to satisfy a certain property. Neither did he present ideals of fields as a distinguished kind of ring. Hilbert's ideals are always ideals in fields of numbers. Moreover, in spite of his earlier direct involvement with the theory of polynomials and his acquaintance with the main problems of this discipline, Hilbert never attempted to use ideals as an abstract tool allowing for a unified analysis of factorization in fields of numbers and in systems of polynomials. This step, crucial for the later unification of the two branches under the abstract theory of rings, was taken more than twenty years later by Emmy Noether. Obviously, the absence of such a step in Hilbert's work is not so much a consequence of technical capabilities, as it is one of motivations: an indication of the nature of his images of algebra, to which the idea of algebraic structures as an organizing principle was foreign.

Hilbert's ideas enormously influenced the course of mathematical research over the first half of the twentieth century. His work in number theory, as well as in the theory of invariants contained many elements that turned out to be fundamental for establishing the structural approach to algebra, but his own conception was different from that. Moreover, his direct influence on students and collaborators was *not* instrumental in bringing about an immediate shift towards the structural approach. One relevant parameter for evaluating this influence is manifest in the doctoral dissertations he supervised (no less than sixty-eight throughout his career). Only four among them dealt with issues directly or indirectly related to Hilbert's first domain of research: invariant theory. Not one of the dissertations deals with problems connected with the theory of factorization of polynomials, although at that same time important works were being published by other mathematicians —such as Emanuel Lasker and Francis S. Macaulay— which elaborated on Hilbert's own ideas. Nor is there any dissertation dealing with topics that later came to be connected with modern algebra —such as abstract fields, or the theory of groups in any of its manifestations— and that knew at the time intense activity throughout the mathematical world. No less remarkable is the fact that, although five among the twenty-three problems that Hilbert included in his 1900 list can be considered in some sense as belonging to algebra in the nineteenth-century sense of the word, none of them deals with problems connected with more modern algebraic concerns, and in particular not with the theory of groups.

The works that finally created the substrate on which the structural approach to algebra emerged (works by Ernst Steinitz, Emmy Noether, and Emil Artin, to mention only the most outstanding) were deeply influenced by mathematicians like Dedekind and Hilbert, and by their ideas as described here: these ideas contained many of the elements needed for constructing algebra under the structural apporach. But the overall conception of algebra within which Dedekind, Hilbert and their nineteenth-century fellow algebraists produced their work was essentially different from the structural one, and we will have to wait until the next generation to see this approach definitely coming into mature existence.

After the publication of *Moderne Algebra* the discipline of algebra grew and developed vigorously within the structural approach and throughout the twentieth century. Many important results were achieved that stressed the advantages of working under that approach. Entire sub-disciplines were reshaped under this new perspective. The most outstanding among them was perhaps algebraic geometry in the hands of Oscar Zariski and André Weil. Likewise, many additional structural textbooks were published thereafter that helped spreading the gospel further on. One of the most influential among them was *Survey of Modern Algebra* by Garret Birkhoff and Saunders Mac Lane, whose first edition appeared in 1941 and which became a cornerstone in the education of generations of mathematicians both in the USA and outside it.

The success of the structural approach in algebra explains to a large extent the adoption of closely related points of view in neighboring disciplines, such as topology and functional analysis. Strongly structurally-oriented views of mathematics as a whole underlie the highly influential work of the French group known as Bourbaki. This view was embodied in their multi-volume treatise *Eléments de mathématique*, whose first installment was published in 1939 and which continued to appear over the next decades, shaping to a large extent the way in which the discipline of mathematics was conceived in many countries around the world, at least between 1940 and 1970.

Finally, the gradual adoption of a structural point of view in various mathematical disciplines raised the natural question whether it would be possible to develop a general, abstract (meta-)mathematical theory that would elucidate the very concept of a structure and in which one could derive general results valid for the various instances of structures arising in different mathematical contexts. The most successful and influential effort in this direction is the one associated with the creation of the theory of categories and functors. Such concepts were introduced by the first time in 1942 in an article by Mac Lane and Samuel Eilenberg and soon thereafter the theory began its autonomous development. Although it by no means became a universal language for mathematics as some of its practitioners may have hoped for, it certainly came to provide a very flexible and general formulation that helped redefine certain important fields, notably algebraic topology and algebraic homology.

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