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Some distributivity-like results in the medieval arithmetic of Jordanus Nemorarius and Campanus de Novara

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Abstract

The present article explores the treatment of some distributivity-like properties in the works of Jordanus Nemorarius and Campanus de Novara. The perspective afforded by this analysis gives rise to some interesting insights concerning medieval attitudes towards the relationship between geometry and arithmetic, in particular as part of the Euclidean tradition. It also sheds interesting light on medieval conceptions about arithmetic as an autonomous discipline requiring its own proper foundation.

Resumen

El presente artículo discute la forma en que propiedades quasi-distributivas de las operaciones aritméticas son presentadas en las obras de Jordanus Nemorarius y Campanus de Novara. La perspectiva ofrecida por este análisis trae a luz algunas observaciones interesantes sobre las actitudes medievales concernientes a la relación entre la geometría y la aritmética, y en particular en lo que corresponde a la tradición euclidiana. La discusión también arroja luz sobre las concepciones medievales de la aritmética como disciplina autónoma que requiere una fundamentación propia y adecuada.

MSC: 01A35

Keywords: Euclid's Elements; Medieval arithmetic; Jordanus; Campanus

1. Introduction

This article explores some "distributivity-like" properties of operations on natural numbers as they surface in the texts of two of the most important mathematicians active in the 13th century, Jordanus Nemorarius and of Campanus de Novara. More specifically it focuses on their role among the basic propositions that these two authors took to be fundamental for their pioneering attempts at providing systematic

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foundations for the corpus of arithmetic knowledge. The main point of interest in this topic is that it sheds light on a peculiar aspect of the processes of border-blurring across the domains of continuous and discrete magnitudes in the medieval Euclidean tradition. These domains were strictly kept apart in the original Euclidean text, but the changing contexts where this text was translated, commented, and relied upon—from late antiquity, via the Islamic tradition, and into the medieval versions—brought about the gradual incorporation of new ideas and led to innovative ways to understand and further develop its mathematical contents. One aspect of these processes is manifest in the way that arithmetic and proto-algebraic ideas became gradually associated with what was once a purely geometrical conception of Book II (Corry, 2013). Another aspect, closely related to it, is that of the ideas related with the "distributivity-like" properties that I want to discuss here.

The article opens with a brief overview on the way that "distributivity-like" results appear in different contexts in the *Elements*. Next it discusses the transformations undergone by these results in al-Nayrīzī's commentary to the *Elements*, dating from the early tenth century. This commentary had considerable impact on medieval authors such as Jordanus and Campanus, whose works are discussed in the last two sections of the article. In a separate article (Corry, 2016), I discuss in greater detail similar, "distributivity-like" results as they appear in works not analyzed here, including all the relevant results appearing in Books II, V and VII of Euclid's *Elements*, and then in works such as those of Abu Kāmil, *Liber Mahameleth*, Fibonacci and Gersonides.

It is relevant to stress that my focus on "distributivity-like" properties is not meant to imply that, in any of the texts discussed here, we find a general, clearly formulated idea of "distributivity" as a fundamental, widely acknowledged, general kind of property underlying the relationship between two basic and also well-defined operations, "product" and "addition". Rather, the ideas discussed here in relation with "distributivity-like" properties developed and consolidated separately, though in parallel and in interaction with each other, as part of a long historical process: product, addition, number, magnitudes and also distributivity. I think that it is historically rewarding to look at them from a common perspective that involves a broad idea of "distributive-like" properties. Accordingly, then, the term is used here as a somewhat loose label that allows a common reference to various kinds of results, rather than as an assertion that this was a clearly conceived, general idea specifically applied in particular cases.

2. Distributivity-like properties in Euclid's *Elements*

I start with a brief overview of how some distributivity-like properties appear in Euclid's *Elements*, and in the first place, in Book II. The first four propositions of this part of the treatise discuss some basic properties of area-formation in rectangles. Proposition II.1, for instance, formulates the following general property[1]:

> **II.1:** If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

The main step in the proof relies on a proposition from Book I, I.34, while making reference to the following diagram, where in the rectangle *BGHC*, the side *BG* equals the given line *A*, and the line *BC* is divided into sub-segments (see Figure 1).

The said proposition I.34 is used to assert that, since *BK* is by construction a rectangle, then *DK* equals *BG* and hence equals *A*. A repeated application of this argument allows concatenating the three resulting rectangles into a single, larger one, and thus to complete the proof.

[1] All the quotations of the *Elements*, including the accompanying diagrams, are taken from Heath (1956 [1908]).

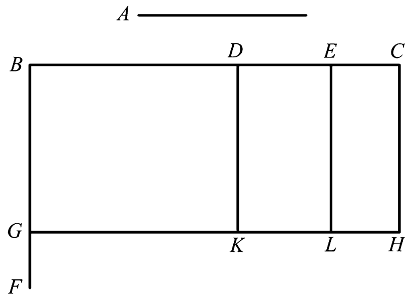

Figure 1. Euclid's *Elements* II.1.

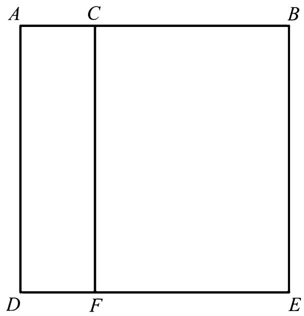

Figure 2. Euclid's *Elements* II.2.

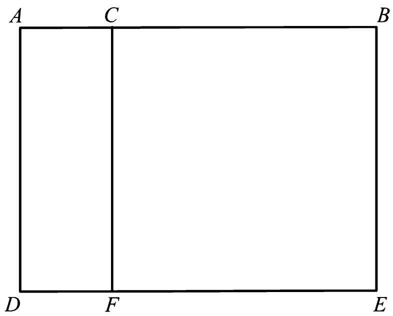

Figure 3. Euclid's *Elements* II.3.

Propositions II.2–II.3 read as follows:

> **II.2:** If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole. (See Figure 2.)
>
> **II.3:** If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment. (See Figure 3.)

In spite of their appearance as particular cases of II.1, Euclid did not prove them by straightforward application of the latter. Rather, the two proofs he offered essentially recapitulate the argument of II.1, but now applied to the particular cases in point. This reflects a more general feature typical of the proofs of the first ten propositions of Book II, namely that none of them relies on a previous one of the same book. Rather they are all proved by directly relying on propositions of Book I alone. The same is true for II.4. Its formulation and diagram are as follows:

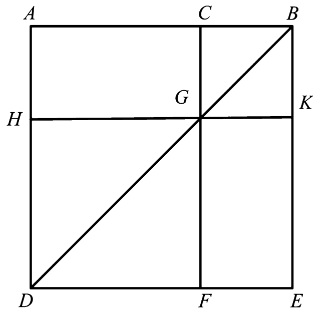

Figure 4. Euclid's *Elements* II.4.

II.4: If a line is cut at random, then the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments. (See Figure 4.)

If we were now to translate the contents of these propositions into modern algebraic symbolism we would get expressions that we can associate with the distributivity of the product over addition. Thus for instance:

I.1: $a \cdot (b + c + d + \ldots) = a \cdot b + a \cdot c + a \cdot d + \ldots,$
I.2: $a \cdot (a - b) + a \cdot b = a^2,$
I.3: $a \cdot (a + b) = a \cdot b + a^2,$
I.4: $(a + b)^2 = a^2 + b^2 + 2 \cdot a \cdot b.$

The misleading potential of such symbolic renderings has been a classical topic in the historiography of ancient mathematics ever since Sabetai Unguru published in 1975 his famous criticism of the "geometric algebra" interpretation (Unguru, 1975). This criticism need not be repeated here. Still, it seems that even if we give up altogether the attempt to present any possible symbolic rendering of them, we can describe these four propositions, without fear of anachronism, as involving distributivity-like properties of two specific geometric operations, namely, area-formation for rectangles over area-addition of them.[2]

In contrast with this, we cannot say that in the arithmetic books of the *Elements* there are propositions that embody—in the same rather straightforward manner as those of Book II do in their own narrow context—distributivity properties of multiplication over addition of integers. Still, there are some propositions in Book VII that do explore distributivity-like properties. Propositions VII.5–VII.8, for instance, focus on the question of "being a part" or of "being parts" of a given number while exploring the issue of equimultiplicity. Proposition VII.5 reads as follows:

VII.5: If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

In his classical edition of the *Elements*, Heath renders this proposition in a manner that makes it appear as involving distributivity, plain and simple, as follows[3]:

[2] For an insightful discussion of Euclid's approach to proving these propositions as separate results see (Saito, 2004 [1985], 157–160).

[3] A similar, though not identical, rendering appears in Itard (1961, 90–97). Taisbak (1971, 40 ff.) gives a completely different kind of symbolic rendering, conceived with the specific purpose in mind of avoiding any possible historical inaccuracy incurred by the use of modern algebraic symbolism. It is well beyond the intended scope of the present article to follow any kind of symbolic approach close to that of Taisbak. Still, it is interesting that, following his own point of view, Taisbak states explicitly (p. 43) that VII.5 and VII.6 "can be interpreted as the *Distributive Law*."

***VII.5*:** $\frac{1}{n}a + \frac{1}{n}b = \frac{1}{n}(a+b)$.

This rendering however, is truly misleading in many respects. For one thing—beyond the general historiographical issues already mentioned for any symbolic rendering of this kind—in Euclid's arithmetic there is nothing like the fraction $\frac{1}{n}$. A closer symbolic approximation, then, could be the following:

***VII.5′*:** If $a = n \cdot b$ and $c = n \cdot d$, then $a + c = n \cdot (b + d)$.

This is not itself without limitations as a faithful rendering of Euclid, but it has two clear advantages. In the first place, whereas Heath's rendering requires the use of fractions, this symbolic representation relies only on the use of whole numbers. In the second place, it retains the "if ... then" format of the original. The main point in introducing a rendering of this kind, however, is to explain in what sense I intend to include the properties explored in VII.5–VII.8 under the heading of "distributivity-like."

Something similar can be said about the first six propositions in Book V. They comprise a self-contained, comprehensive discussion of results concerning equimultiplicity of continuous magnitudes. Their purpose is to provide basic results that are used to develop in full the Eudoxian theory of ratios and proportions, but ratios and proportions as such are not mentioned in the six propositions. Rather, they refer only to the magnitudes and to their multiplicities. Let us consider the example of V.1, which reads as follows:

> **V.1:** If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitude equal in multitude, then, whatever multiple one of the magnitude is of one, that multiple also will all be of all.

Also in this case, Heath's symbolical rendering sets the stage for a possible interpretation of the six propositions as general statements of distributivity laws. In this example it involves the following expression:

***V.1*:** $m \cdot a + m \cdot b + m \cdot c + \ldots = m \cdot (a + b + c + \ldots)$.

But once again—beyond the general historiographical difficulties already mentioned—there are specific issues to consider in this particular rendering. Indeed, the "multiple" of a given magnitude M is not, for Euclid, the outcome, $n \cdot M$, of a binary operation, namely, multiplying a given magnitude M by the number n, as Heath's rendering suggests. Nor is it the result, $M + M + M + M + \ldots + M$, of successive steps of binary additions of the magnitude to itself n times. Rather, it is more of an "accumulation" or of a "gathering together" of a *multitude* of instances of the said magnitude: $M, M, M, M, \ldots M$. Heath's rendering also does not reflect the "if … then" style of formulation of the propositions. Thus, a possible symbolic rendering that would remain closer to Euclid in at least this formal and important respect would be the following:

***V.1′*:** If $a' = m \cdot a, b' = m \cdot b, c' = m \cdot c, \ldots$ then $a' + b' + c' \ldots = m \cdot (a + b + c \ldots)$

Again, having stated its limitations, I take a rendering of this kind to justify my intention to include the properties explored in V.1–V.6 under the heading of "distributivity-like". As we will see below, medieval treatises where distributivity-like properties were discussed mixed arguments taken from Book II with those taken from these specific propositions of Book V.

3. Al-Nayrīzī

Heron of Alexandria wrote a *Commentary* of the *Elements* at the end of the first century A.D. It contained alternative proofs for several propositions in Book II. The approaches that both Euclid and Heron followed

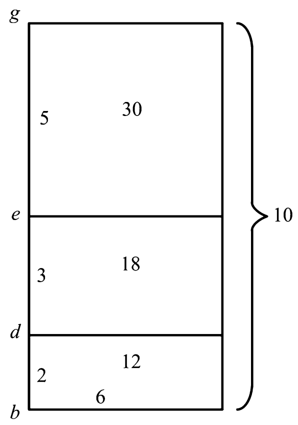

Figure 5. Al-Nayrīzī's diagram for Euclid's *Elements* II.1.

in their respective proofs were purely geometric yet quite different from each other. The former can be described as "constructive" whereas the second as "operational" (Corry, 2013, 652–654). Heron's text is lost but his ideas are known to us via al-Nayrīzī's commentary to the *Elements*, dating from the early tenth century. This is one of the earliest such commentaries to be written in Arabic, and it significantly influenced the way that later readers understood some parts of the treatise (Hogendijk, 2010).

In the context of Book II, together with his report on Heron's ideas, al-Nayrīzī added numerical examples of his own, which were meant to illustrate the meaning of each proposition. This kind of numerical interpretation could have hardly been accommodated within the specific approach followed in Euclid's proofs since they were, as I said, "constructive". To the contrary, they found a more natural place within Heron's because, while geometrical, they were "operational" (Corry, 2013, 661–662).

Heron asserted that II.1 is the only one among the fourteen propositions that "cannot be proved without drawing a total of two lines". For the remaining thirteen propositions, he stated that "it is possible that they be demonstrated with the drawing of one sole line" (Curtze, 1899, 89).[4] There is no report on Heron's argument for II.1, and we may assume that it added nothing to Euclid's original. Al-Nayrīzī's numerical example for II.1 (p. 88), in turn, is embodied in Figure 5.[5]

Al-Nayrīzī also reported on Heron's proofs for the other propositions in Book II. The most important feature of these proofs is that, unlike Euclid's original proofs, they are not separate proofs that rely, each of them, directly on results of Book I. Rather, after II.1 is proved based on results of Book I, the remaining proofs rely on that of II.1, and then on other propositions from the same Book II, as gradually proved. Thus, for instance, II.2 and II.3 appear here naturally as particular cases of II.1. Then, II.4 appears as directly derivable from II.1, via the other two (p. 92).

But al-Nayrīzī's own contribution went much further than just illustrating the propositions of Book II with numerical examples: he took the further step of incorporating into the arithmetical books of the *Elements* arithmetic versions of propositions II.1–II.4. These arithmetic versions appear as commentaries added to a result in Book IX that, remarkably enough, does *not* appear in Euclid's original version of the arithmetic sections of the *Elements*. I call this proposition here AN-IX.16. As a matter of fact, AN-IX.16 is a generalization of Euclid's IX.15, which reads as follows:

> **IX.15**: If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

[4] Unless otherwise stated, translations from Latin are mine.
[5] In his commentaries to the text of Al-Nayrīzī (Curtze, 1899), Curtze added algebraic renderings to each proposition.

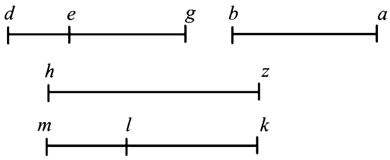

Figure 6. Al-Nayrīzī's diagram for the arithmetic version of Euclid's *Elements* II.1.

Al-Nayrīzī's formulation is similar to this, but rather than three, it speaks about the addition of an arbitrary number of numbers.[6] We can understand this slight, but nonetheless significant modification against the background of a more flexible conception of number that evolved in Islamicate mathematics, along a less rigid separation of discrete and continuous magnitudes. Less rigid, it must be stressed, but not altogether inexistent. This move, at any rate, had a direct, visible influence on later medieval treatises, and specifically on Jordanus and Campanus as we will see below.

The proofs that al-Nayrīzī provided for the arithmetic results presented in the commentaries to AN-IX.16 are typical of those appearing in Euclid's arithmetic proofs. This is to say, in the diagrams (Fig. 6, below), lines serve to indicate the numbers involved in the proof, but they are not used to produce any relevant *geometric* constructions. In particular, multiplication is never represented in the Books VII-IX of the *Elements*, either in its original or in al-Nayrīzī's version, as area formation, in contrast to what is the case in Book II (Figures 1–4, above).

Thus, for instance, Al-Nayrīzī's diagram for the arithmetic version of II.1, appearing in his commentaries to AN-IX.16 (p. 204), is shown in Figure 6. The line hz represents the product of ab by gd, whereas kl represents the product of ab by ge and ml that of ab by ed. The proposition states that hz equals km. The proof proceeds simply by spelling out each multiplication as the number of units by which a number measures another. Thus, "gd measures (*numerat*) hz by as many units as there are in ab whereas ge measures kl by as many units as there are in ab and ed measures ml by as many units as there are in ab". From here the conclusion is reached that the addition (*conjunctio*) gd measures km by as many units as there are in ab, and hence the number km equals the number hz. Translating back into "multiplication", al-Nayrīzī concludes that the area that is obtained from ab and gd equals the addition of the two areas that are obtained from ab and ge and ab and ed. And this is what we wanted to prove.

Thus we see that, while attempting to incorporate these results into the corpus of arithmetical knowledge displayed in the *Elements*, al-Nayrīzī nevertheless abode by the basic separation of realms. He did not import into the arithmetic books the kind of geometric reasoning with continuous magnitudes used by Euclid in Book II, but rather developed a proof that followed Euclid's own constraints for dealing with discrete quantities. Al-Nayrīzī dealt here with discrete magnitudes and his proof was based on implicitly rearranging, according to the need, the instances of the magnitudes that appear in the said multitudes. As we will see, such rearrangements are performed explicitly in the work of Campanus as well, as the basis of some of his arguments.

Al-Nayrīzī also formulated arithmetical equivalents of II.2–II.3 and it is interesting to notice how he proved them. Recall that in his (geometric) version of Book II, he had relied on II.1 for proving these two geometric propositions, just as Heron had done. But now, in proving these arithmetic versions of II.2–II.3, he did *not* rely on his own arithmetic version of II.1. Rather he rehearsed the same (arithmetic) argument that he introduced for his arithmetic version of II.1, adapting it to the particular cases of II.2–II.3. And this is

[6] "*Si fuerint numeri quotlibet continue proportionales in sua proportione minimi*, ...". Curtze cites the proposition in a footnote (p. 204), without mentioning the discrepancy with the Euclidean original. As we shall see below, Campanus followed al-Nayrīzī's formulation.

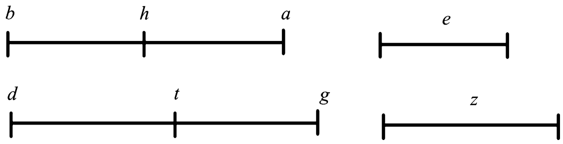

Figure 7. Al-Nayrīzī's diagram for Euclid's *Elements* V.1.

what Euclid had done, *mutatis mutandis*, for proving the geometric versions of the same two propositions. And then, in the arithmetic version of II.4 he followed the same approach, and proved it by relying on II.2 (pp. 205–207). Thus, it is evident that for al-Nayrīzī it was important to stress the autonomous, purely arithmetic characters of the propositions that were presented as commentaries to AN-IX.16.

Also al-Nayrīzī's comments to propositions V.1–V.2 are interesting. First, concerning V.1 (already mentioned above), he indicated a possible difficulty in the argument of the proof (pp. 169–170). In order to see what he had in mind, consider the accompanying diagram in Figure 7.

Here ba, dg represent equimutiples of e, z respectively, and the argument of the proof requires that the each of the latter be cut from each of the former, respectively. Now, in the simplest cases, this raises no difficulties. For example, if the two given magnitudes ab and gd are lines, he wrote, then, in order to do so, we can invoke Euclid's I.3 ("Given two unequal straight lines, to cut off from the greater a straight line equal to the less"). In the case where the magnitudes are arcs, al-Nayrīzī invoked Book III as providing the necessary justification. Most likely he had in mind a combination of propositions such as the following two:

> **III.27:** In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

> **III.34:** From a given circle to cut off a segment admitting an angle equal to a given rectilinear angle.

Also for the case when the magnitudes are arcs al-Nayrīzī declared that no problem arises, but he did not explain why. It is also plausible that his implicit justification for this case might have been related, via arcs of circles, to the same two results of Book III, in conjunction with the following one:

> **VI.33**: In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

But for the case when the magnitudes involved are bodies, al-Nayrīzī indicated that the necessary operation of subtraction becomes "impossible" ("... *tunc illud erit impossibile*."). Nevertheless, he asserted, the existence of multiples is assumed in this case, only in order to imagine that if the number of times that e measures ab is two, then the number of times that z measures gd is also two, or if it is half of this then also z is half gd, and so on for any multiplicity whatsoever.

Thus, al-Nayrīzī considered proposition V.1 as embodying several different, but always *specifically geometric* situations, each of which required its own kind of justification. In other words, he was not thinking of "magnitudes" as a completely general concept on which we can argue in abstract terms, without specific justification for each case. Properties of equimultiplicity, so it seems, were for him differently rooted in basic properties specific to each kind of magnitude that can be considered.

His comments on V.2 are somewhat cryptic and we can at best conjecture what was that he had in mind. Thus, Al-Nayrīzī asserted that "there is nothing at all except the order of the branches of knowledge, of which the first is arithmetic, which is about numbers, and after which comes geometry. The proposition therefore demonstrates the basic, necessary principles which we will discover in this theory" (p. 170). As

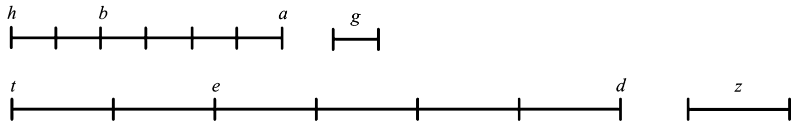

Figure 8. Al-Nayrīzī's diagram for the arithmetic version of Euclid's *Elements* V.2.

a possible interpretation of what he had in mind here, recall the original formulation of the proposition, which reads as follows:

> **V.2:** If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

The "basic principle" to which al-Nayrīzī's was referring in this context has to do with an argument he presented in relation with the figure he used for the proof, see Figure 8.

And his argument is this:

> Once we know that g measures ab according to the number of times that z measures ed, then the times that g measures ab and that it measures bh equal the number of times that z measures de and z measures et. Hence, the enumeration of multiples ah equals the enumeration of multiples dt, and this is what we wanted to prove.

As we just saw, in his treatment of V.1 al-Nayrīzī found it relevant to speak about the meaning of the proposition with respect to various kinds of geometrical magnitudes. Now in discussing V.2 he invoked the importance of "the order of the branches of knowledge" and then limited himself to a general argument presumably valid for all kinds of magnitudes. Perhaps he meant to say that it is not necessary to discuss the various cases separately because his argument covers "the basic, necessary principles that we will discover in this theory". Admittedly, this conclusion is somewhat conjectural. Much less can we know how later readers interpreted it, or if they paid attention to this remark at all.

4. Jordanus Nemorarius

I jump now to the 13th century, and to one of the most remarkable figures of the time in matters mathematical, Jordanus Nemorarius. We are interested here in his book *Arithmetica*, which comprises a pioneering attempt to provide systematic foundations for the arithmetic of the natural numbers seen as an autonomous field of knowledge. Jordanus' declared intention was to achieve here what Euclid had done for geometry, in all that concerns the derivation of the body of arithmetic, starting from definitions, postulates, and common notions. In doing so, one of his main aims was to avoid reliance on geometrical concepts or results of any kind when putting forward his project (Corry, 2013, 684–689).

A detailed analysis of the actual axiomatic structure of the *Arithmetica*, and the extent to which Jordanus achieved his intended aim has never been undertaken thus far, to the best of my knowledge. It is, however, beyond the scope of the present paper. What is clear on the face of it is that Jordanus' attempt to systematically avoid possible geometric reasoning in his presentation of arithmetic was much more explicit and thoroughgoing than anything similar found in earlier texts. The way in which distributivity-like properties appear in Jordanus' treatise, in particular, is detailed and original, and certainly worth of close examination.

In the opening section of *Arithmetica*, Jordanus proved five results that I want to characterize here under the heading of "distributivity-like". They are the following[7]:

[7] For easiness of reference I use here a numeration of the propositions which does not appear in the original.

A-I.4: If a first number is the same part of a second number that a third is of a fourth, then the first and the third are the same part of the second and the fourth than the first is of the second.

A-I.5: If a first number is the same parts of a second number that a third is of a fourth, then the first and the third are the same parts of the second and the fourth than the first is of the second.

A-I.6: If as many numbers as you please are equimultiples of as many other numbers, then the sum of the numbers is the same multiple of the sum of the other numbers.

A-I.9: That which is obtained by multiplying any number by as many as one pleases equals that which is obtained by multiplying the same [number] by their combination [i.e., their sum].

A-I.10: That which is obtained by multiplying as many numbers as one pleases by some number equals that which is obtained by multiplying their combination [i.e., their sum] by the number.

One way to see in why it makes sense to speak of these five statements as embodying distributive-like properties is by comparing them with those appearing in the *Elements*. It is quite straightforward to establish the following parallels:

Jordanus' *Arithmetica*	Euclid's *Elements*
A-I.4	VII.5
A-I.5	VII.6
A-I.6	V.1
A-I.9	II.1
A-I.10	II.1*

Notice that the first three of them, like their Euclidean counterparts, are statements of the form "if ... then". The last two embody properties that resemble left- and right-distributivity of the product over addition for numbers. Also, notice that whereas the counterparts of the first two of Jordanus' A-I.4, A-I.5 (i.e., VII.5–VII.6) had originally been formulated by Euclid for numbers, the one which is parallel to A-I.6 (i.e., V.1) was a proposition about ratios. The one which can be seen as parallel to both A-I.9 and A-I.10 (i.e., II.1) was about areas of rectangles. Each of them, of course, was formulated in the *Elements* in accordance to the topic of the Euclidean book where they were treated. Likewise, Euclid's proofs relied on different kinds of arguments corresponding to the various different contexts. In stark contrast to this, Jordanus presented all these five propositions in the introductory section of his treatise as purely arithmetic properties satisfied by operations with numbers. And yet, the proofs of these propositions, while fully arithmetic in essence, retain to some extent the taste of the different context where Euclid had originally formulated them. Let us see some details of this.

In the first place, it is helpful to render the five propositions in some kind of symbolic notation that stresses their character as possible embodiments of "distributivity-like" properties. One way to do it is the following:

A-I.4: If $a = n \cdot b$ and $c = n \cdot d$, then $a + c = n \cdot (b + d)$,
A- I.5: If $m \cdot a = n \cdot b$ and $m \cdot c = n \cdot d$, then $m \cdot (a + c) = n \cdot (b + d)$,
A-I.6: If $a' = m \cdot a$, $b' = m \cdot b$, $c' = m \cdot c, \ldots$ then $a' + b' + c' \ldots = m \cdot (a + b + c \ldots)$,
A-I.9: $a \cdot b + a \cdot c + a \cdot d + \ldots = a \cdot (b + c + d + \ldots)$,
A-I.10: $b \cdot a + c \cdot a + d \cdot a + \ldots = (b + c + d + \ldots) \cdot a$.

As usual, one must exercise great care when using modern symbolic notation for making sense of ancient and medieval mathematics. In the specific cases we are considering here, special attention must be directed at the kind of operations intended in A-I.6, on the one hand, and in A-I.9, A-I.10, on the other hand. They are not the same, even when they look similar when rendered symbolically. For one thing, in A-I.9, A-I.10 we have a binary operation with numbers (the product of a number by a number), which is an arithmetic version of a proposition that Euclid had proved for areas. In A-I.6, on the other hand, the "operation" in point is the "accumulation" or "gathering together" of units that we also find in Book V of Euclid (as explained above) but not in the context of Book II.

One direct consequence of this difference is that there is nothing like a reciprocal statement for A-I.6, of the kind that A-I.9, A-I.10 can be said to be of each other. The "coefficient" n appearing in the symbolic rendering of A-I.6 is not a number multiplying the following numbers, but rather an indication of the amount of instances gathered together. And yet, this important difference would have been much more significant in the context of Euclid's *Elements* than in that of Jordanus' *Arithmetica*. By the time of the latter, the gradual process whereby arithmetic ideas had come to be increasingly associated with the results of Book II had strongly weakened the separation across domains of continuous vs. discrete magnitudes. The very fact that Jordanus chose to gather this list of elementary propositions as the common base for developing the entire corpus of arithmetic attests for the conceptual "proximity" that he accorded to them. At this stage of the historical process, I think, the close resemblance that the symbolic rendering makes apparent between these five operations involves more than just a misleading anachronism.

I am not claiming, to be sure, that Jordanus is displaying something like a full awareness of distributivity as a general property of two abstract operations of multiplication and addition on any type of magnitudes, and much less that there is some kind of underlying idea here that is directly translated into the kind of symbolic rendering I suggested above. My claim is just that the ongoing processes of change in the way that numbers and continuous magnitudes were used in the Euclidean tradition and the blurring of borders across the domains is reflected also in a parallel, ongoing awareness to a proximity between all these properties within the domain of arithmetic. In addition, I think that the said proximity can be characterized in terms of their being "distributivity-like" properties.

I think that in this context it is also revealing that Jordanus did *not* include counterparts to Euclid's VII.7, VII.8, which run parallel to VII.5, VII.6 but involve *subtraction* rather than addition. The arithmetic renderings of the propositions in Book II are always about additions of some kind, given that in Book II the areas considered are always concatenated and not subtracted. So, distributive-like properties involving subtraction (i.e., putative counterparts of VII.7, VII.8) would find no natural place in Jordanus' overall presentation, and indeed, they do not appear in his *Arithmetica*.

It is likewise interesting to notice the absence of anything parallel to Euclid's V.2 (already stated above). Notice that if we were to render symbolically the contents of this proposition in a way similar to that of the other ones discussed above, we would obtain the following:

***V.2′*:** If $a = m \cdot b$, $c = m \cdot d$ and $e = n \cdot b$, $f = n \cdot d$, then, while $a + e = p \cdot b$, we also have $c + f = p \cdot d$. In this symbolic rendering one sees that $p = m + n$.

Acknowledging here any proximity with distributive-like properties would mean that it applies also to the "coefficients" (which as I just said, are *not* actual coefficients). It would suggest going beyond the conceptual proximity that, I think, underlies the common treatment of the five statements above.

Also Jordanus' decision to include the pair A-I.9, A-I.10 (whereas the Euclidean geometric counterpart, I.1, had involved only a "left-distributive-like" property of area-formation) can be understood against a similar background of growing conceptual proximity between domains of discourse. Indeed, there is much more to this than just the lack of a similar pair in Euclid. In order to see why, we need to go into the details of some of the proofs.

Jordanus proved propositions A-I.4, A-I.5 very much as Euclid had done for VII.5, VII.6 in the *Elements*. That is to say, in the proofs he counted units and compared. His proof of A-I.6, on the other hand, differs in interesting ways from its Euclidean counterpart, V.1. This should come as no surprise, of course, given that Euclid was working in Book V with continuous magnitudes whereas Jordanus was working here with numbers. Euclid had counted the multiplicity of times that certain magnitudes measure other magnitudes. Jordanus, in turn, reasoned in a way that can be described as "recursive", in the following manner:

(a.1) if we were adding two numbers, then the proposition would be true because of A-I.4 (i.e., Euclid's VII.5),
(a.2) if we further added a third number, then the argument would still hold valid, since we might consider the third number as being added to another number, i.e., the sum of the first two,
(a.3) and the same would hold for the subsequent numbers, which can always be seen as being added to the result of the partial addition of all previous ones, until the entire addition is completed.[8]

Jordanus was thus relying on a result of Book VII for his argument. Euclid's proof of the similar result in Book V, of course, could not have done the same. Notice also that, since the operation involved is not one of binary adding in a more modern sense but rather accumulation of units, Jordanus' kind of "recursive" argument—which is not typically found in other medieval texts on arithmetic, if at all—works in an intuitive manner and he provides no separate justification for it.

Now, while propositions A-I.4–A-I.6 deal with multiplication in the sense of accumulation or gathering together of units (as intended in Book V of the *Elements*), the pair A-I.9, A-I.10 is meant to show that the distributivity-like properties, proved in Book II in a purely geometric way, remain valid in the arithmetic context and can be proved in purely arithmetic terms. Moreover, in these two propositions, Jordanus was after a more transparent derivation of the property for a multiplication of numbers, which would be parallel to area formation in the geometric context. This is why he counted units only in the proofs of A-I.4–A-I.6 (as al-Nayrīzī had previously done), but followed a more detailed approach for A-I.9, A-I.10. In Book II, we have an operation on two entities of the same kind (lines), and also in A-I.9–A-I.10 we multiply entities of the same kind (numbers).[9] On the contrary, in A-I.4, A-I.5 we do not multiply in any way entities of the same kind (numbers) with each other, but rather gather together a collection of certain numbers.

And this is also what explains why Jordanus found it convenient to include in this section a "right-distributive-like" counterpart (A-I.10) of the "left-distributive-like" property embodied in A-I.9. Indeed, having the added numbers on the right or on the left of the multiplication are for Jordanus two different situations. In the cases considered in A-I.4–A-I.6, there is no meaning to the idea of "repetition" appearing either the left or to the right as it would seem to appear if we represent it symbolically. "Repeated addition" is just that. There is only one number involved here that is repeatedly added. The multiplicity n that appears in the symbolic rendering does not actually exist in the rhetoric formulation. There is no such number in the same sense as those that are added. It is certainly not an entity that can be represented symbolically in the same manner as we may allow ourselves to represent the multiplicand symbolically. But propositions A-I.9, A-I.10 represent something different from A-I.4–A-I.6 with this "multiplicity" that is not a numerical multiplier. For the same reasons, they embody two arithmetically different ideas: a "number" is multiplied by "as many numbers as one pleases", and "as many numbers as one pleases" are multiplied by "a number".

[8] Busard (1991, 67): "Eadem erit ratio tertio connumerato cum composito priorum. Similiter etiam sequente semper cum composito ex precedentibus composito usque ad ultimum computato patebit demonstratio."

[9] Of course, there is a difference: in the geometric realm multiplication of two lines yields an area (i.e., an entity which is of a different kind from those that were operated upon) whereas in the arithmetic domain addressed here by Jordanus, the multiplication of two numbers yields a third number. This does not contradict, of course, Jordanus' intention to handle the latter case as a true multiplication of two entities of the same kind.

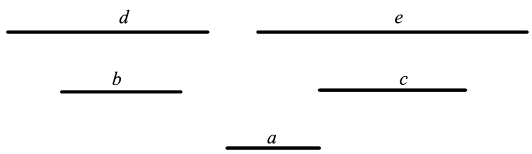

Figure 9. Jordanus' *Arithmetica* A-I.9.

Without wanting to make too much of vocabulary, one cannot fail to see that also at the level of terminology there is an interesting difference. Whereas in the first three propositions (A-I.4, A-I.5, A-I.6), Jordanus spoke of "being part or parts of another", as was typically the case concerning results in Books V and VII of the *Elements* (" ... *quod* a *et* c *erunt partes* b *et* d *quote fuerit* a *in* b."), in propositions A-I.9, A-I.10 multiplication is referred to with the more geometrically-laden terminology usually adopted in relation with Book II throughout the tradition of the Euclidean translations ("*Quod fit ex ductu* ... *in* ...").[10]

In a similar vein, one may notice a remarkable difference in terms of the diagrams between the first three and the last two. First of all, only for A-I.9 and for none of the preceding ones, Jordanus used a diagram in the proof. But more importantly, the crucial point of the logical structure of the entire preliminary section of the *Arithmetica* lies in the fact that Jordanus proved A-I.6 based on an original recursive argument. Subsequently, for A-I.9, he did no more than hint to A-I.6, while rephrasing the result in the language adopted in the two propositions. Finally, he proved A-I.10 by invoking commutativity of the product in general, a result proved in the *Elements* as VII.16. Jordanus essentially repeated the Euclidean formulation in A-I.8, as follows:

> **A-I.8:** If two numbers are multiplied alternately, the same number is obtained in both cases.

It is relevant to look at some of the details of these proofs. First, the proof of A-I.9, which is accompanied by the diagram shown in Figure 9. Here a is multiplied by b and by c to produce d and e, respectively, and the propositions states that the addition of d and e is obtained by multiplying a by the sum of b and c. All that is said in the proof is that "it is evident, by definition, that b counts d according to a, and that c counts e according to the same. Thus, by the sixth [i.e., A-I.6], it is easy to prove this proposition."

The text of the proof of A-I.10 is rather cryptic, but the role of A-I.8 is explicitly stressed. This is what Jordanus wrote (p. 68):

> By [A-I.8], that which is made by multiplying each of them by it, equals that which is made by multiplying it by them all. Likewise that which is made by multiplying the sum of them by it equals that which is obtained from it by the sum. The argument is therefore evident from what was previously said [i.e., by A-I.9].[11]

We may try to understand this argument by rendering it symbolically (with due care) and by conjecturing those steps that Jordanus had in mind and did not write explicitly in the text. I suggest the following:

(b.1) By A-I.8: $b \cdot a + c \cdot a + d \cdot a + .. = a \cdot b + a \cdot c + a \cdot d + \ldots$
(b.2) Likewise [i.e., by A-I. 8]: $(b + c + d + \ldots) \cdot a = a \cdot (b + c + d + \ldots)$
(b.3) But by A-I.9: $a \cdot b + a \cdot c + a \cdot d + \ldots = a \cdot (b + c + d + \ldots)$

10 In saying this I do not have in mind, by any means, a philological analysis of the terms used in each case. I am just pointing out the visible fact that one kind of term is used in one case and a different kind in another case.

11 "*Quod enim fit ex illis in illum equum est ei quod fit ex ipso omnes illos per antepremissam. Itemque quod fit ex composito in illum equale ei quod ex illo in compositum. Per premissam itaque patet argumentum.*"

(b.4) Hence, from (b.1) + (b.3): $b \cdot a + c \cdot a + d \cdot a + \ldots = a \cdot (b + c + d + \ldots)$
(b.5) And from (b.2) + (b.4): $b \cdot a + c \cdot a + d \cdot a + \ldots = (b + c + d + \ldots) \cdot a$ **Q.E.D**

Thus, commutativity of multiplication of individual pair of numbers is used here in order to show that right-hand multiplication of numbers distributes over the addition.

It is worth mentioning, in order to conclude this part of the discussion, that in addition to these five propositions, some of the subsequent ones in the preliminary section of the *Arithmetica* embody additional cases of distributivity-like properties of various kinds. These are, as a matter of fact, versions of II.1–II.3, which generalize or provide particular cases of the previous ones. They are proved by relying on the previous ones, and particularly on A-I.9. Thus, for instance, the following three (of which I just give the symbolic rendering):

A-I.11: $(a + b + c + \ldots) \cdot (p + q + r + \ldots) =$
$a \cdot p + a \cdot q + a \cdot r + \ldots + b \cdot p + b \cdot q + a \cdot r + \ldots + c \cdot p + c \cdot q + c \cdot r + \ldots$
A-I.13: If $a = b + c + d + \ldots$ then $a \cdot a = a \cdot b + a \cdot c + a \cdot d + \ldots$
A-I.14: If $a = b + c$ then $a \cdot b = b \cdot b + b \cdot c$

We can summarize this section, then, by stressing once again that Jordanus' conscious attempt to provide a rigorous presentation of arithmetic, in a way and to an extent not found in any previous treatises, led him to make a clear distinction between "repeated addition of numbers" and "multiplication of two numbers". At the same time, however, he presented those ideas as closely related. A main focus of attention in pursuing that distinction appears in relation with distributivity-like results, and a possible reason for this is that in those treatises where he learned his arithmetic, he did not find a satisfactory treatment of such results.

5. Campanus

Campanus of Novara (*c.* 1220–1296) completed his Latin version of the *Elements* sometime between 1255 and 1259. His treatment of Book II does not differ from the standard, Euclidean one found in other medieval versions of the treatise (Corry, 2013, 689–692). His treatment of distributivity-like results in the framework of Books V and VII, however, is highly original and it deserves special attention.

In the preliminary section to Book V, Campanus introduced lengthy additions and comments. He did not hesitate to explain to his readers what, in his view, was in Euclid's mind when writing this or that definition (and he did the same in his comments to some of the propositions). He also commented on the highly difficult character of the theory of proportions as presented in Book V, while stressing explicitly that these difficulties arise mainly from the need to deal, within one and the same framework, with irrational as well as with rational ratios (Busard, 2005, 173–175). Campanus devoted some efforts to discuss, echoing Jordanus, the significance of Euclid's double treatment of proportions, once for continuous magnitudes and once for numbers (Corry, 2013, 692).[12]

In order to help the reader in following and understanding the arguments of the proofs in the arithmetic books of the *Elements*, Books VII–IX, Campanus associated numbers to the segments appearing in all of the diagrams. Interestingly, he followed the same practice for the diagrams in Book V, which is meant to be about magnitudes in general rather than about discrete magnitudes. Jordanus, as we saw, had followed a similar approach, and it was quite natural that he did so as part of his treatment of results in the arithmetical books. It seems much less natural, however, to find this done in the case of Campanus, given his explicit stress on the essential difference between handling proportions that involve continuous magnitudes and

12 Campanus' discussion of proportions also raises some additional issues concerning both textual and conceptual difficulties, but they are beyond the scope of this article. See Rommevaux (2007).

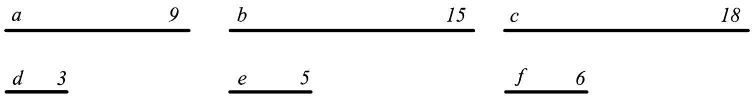

Figure 10. Campanus' diagram for Euclid's *Elements* V.1.

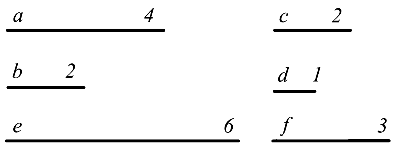

Figure 11. Campanus' diagram for Euclid's *Elements* V.2.

those that are purely arithmetic. Let us see how this appears, for instance, in the diagram accompanying V.1 (p. 177), which is shown in Figure 10.

Here the three magnitudes a, b, c are said to be equimultiples of d, e, f respectively, and the proposition states that $a + b + c$ is the same equimultiple of $d + e + f$ as a is of d. The numbers appearing in the diagram are not even mentioned in the specification or in the proof, but one can imagine that they may have helped the reader follow the argument.[13]

The proof follows a rather original approach based on an inductive argument of sorts on the multiplicity involved. Thus, Campanus first considered the case in which the magnitudes are "respectively equal" ("*quod si singule singulis sint equales*"). By saying this, he meant the case in which $a = d$, $b = e$, $c = f$, whence the proposition is evidently true on the basis of the common notion: "*si equalibus equalia addantur, tota quoque erunt equalia*". But what happens if we have actual multiples, as in the diagram above? Here Campanus reasoned as follows:

(c.1) Separate each of the greater magnitudes into the lesser magnitudes that compose them.
(c.2) Take the first component of a, the first of b and the first of c, and add them together. This addition, says Campanus, equals that of d, e, f, and he cites here a different common notion as the ground for reaching this conclusion: "*que eidem sunt equalia, inter se sunt equalia*". One can guess that the meaning is that each of the parts composing a number, say a, is equal to any other part of the same number and hence each is equal to d. And since this is the case for the three magnitudes, a, b, c, the first part of the arguments holds.
(c.3) A further step is still needed in order to conclude the proof, namely, that the same reasoning applies to the second part of each of the magnitudes ("*Similiter quoque aggregatum ex secundis partibus quantitatum* $a, b, c \ldots$"), and so forth as many times as d is contained in a.

The proof of V.2 is not really different from Euclid's, except for the diagram. Campanus' diagram also associates numerical values with each magnitude, as shown in Figure 11. Notice that the magnitudes on the left need not be of the same kind as those on the right. This point is explicitly mentioned neither by Euclid nor by Campanus, but Euclid did not include any numbers in his diagram, and the fact that they do appear in Campanus' diagram makes this fact even less evident. Also unlike Euclid, but as in most other medieval texts of the kind, Campanus provided specific justification only for some of the steps in the proof.

Thus, the reader is told that the number according to which b is contained in a (let me call it here n) equals the number according to which d is contained in c, and likewise, the number according to which b is contained in e (let me call it here m) equals the number according to which d is contained in f. Then, in

[13] In the editorial comments of (Busard, 2005) there are no hints to the possibility that the diagrams in general, and the addition of numerical values in particular, should be attributed to anyone other than Campanus himself.

order to move to the next step he invoked one of the "common notions" formulated in the opening section of his book: if equals are added to equals, the results are equals. Hence, the number according to which b is contained in the addition of a and e (let me call it here $n + m$) equals the number according to which d is contained in the addition of c and f.

Also Campanus' treatment of distributivity-like properties in Book VII deserves attention. In fact, his entire treatment of arithmetic in the framework of the *Elements* is highly interesting and original, and this is just one aspect of his more general approach. Very much as in Jordanus' *Arithmetica*, and clearly under its overt influence (often to the extent of verbatim repetition), the opening section of Book VII in Campanus' version of the *Elements* is built as an attempted axiomatic presentation that runs parallel to that of Book I for geometry: twenty-four definitions, four postulates (*petitiones*), and ten "common notions" (*communes animi conceptiones*). As in the case of Jordanus, a detailed analysis of this intended axiomatic presentation and how it materializes in the three arithmetic books of the *Elements* would be a highly appealing task. It is, however, beyond the scope of the present article (but see Rommevaux, 1999).

Already in the introductory section to Book VII we can notice that Campanus devotes focused attention to distributivity-like properties. Thus, one of his common notions, which is fundamental to the attempted systematic foundation of arithmetic, states that if the unit is multiplied by any number or if a number is multiplied by the unit, then the result is the number itself (Busard, 2005, 231). As a matter of fact, also in Jordanus we find a similar statement (Busard, 1991, 65). But unlike Jordanus, Campanus added three additional ones, and I include them here under the category of "distributivity-like". They are the following:

- Any number that measures two numbers, measures also their sum.
- Any number that measures some number, measures also any number measured by it.
- Any number that measures the whole and the deducted, measures also the remainder.

Intrinsically related with the attempt to provide an axiomatic foundation for arithmetic is the idea of the autonomy of such a body of knowledge vis-à-vis the other parts of mathematics presented in the *Elements*. Campanus consistently stressed this issue throughout the arithmetic books, and in particular he stressed the autonomy of proofs in Book VII vis-à-vis those of Book V. Campanus explained that the "*propria principia*" of the two books are different and, hence, corresponding propositions should be proved separately, and based on those specific principles alone in each case (Corry, 2013, 690). In relation with the proof of VII.5, for instance, we find the following statement (p. 235):

> Euclid wanted that the arithmetical books would not have to rely on the previous ones, but rather that they would stand by themselves. Results that he proved in the fifth book for quantities in general he proved here for numbers in this fifth of the seventh.

When examining the proofs in some detail, however, we notice that in some cases this autonomy did not go beyond repeating, while fully rewording for numbers, an argument already presented in Book V.[14]

In some interesting cases, Campanus did work out the specificity of the arithmetic situation involved and such cases afford some insights into his peculiar conceptions. Such is the case with his treatment of VII.6, which, reads as follows:

> **VII. 6:** If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

[14] In other Latin versions of the *Elements*, instead of such a repetition, often there is just a direct reference to a corresponding proposition in Book V. See Busard (2005, 560).

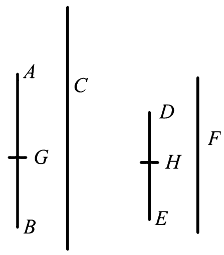

Figure 12. Euclid's *Elements* VII.6.

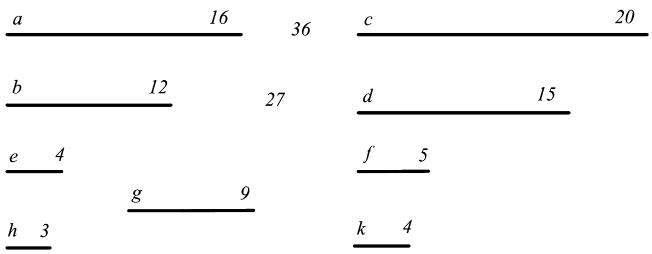

Figure 13. Campanus' diagram for Euclid's *Elements* VII.6.

Symbolically rendered it states that:

***VII.6*$'$:** If $m \cdot a = n \cdot b$ and $m \cdot c = n \cdot d$, then $m \cdot (a + c) = n \cdot (b + d)$.

Euclid's diagram, as known to us, is shown in Figure 12.

The basic argument of Campanus' proof is not essentially different from Euclid's original, but it is presented in a completely different manner. This difference can be immediately appreciated by looking at the accompanying diagram, typical of Campanus, which is shown in Figure 13. The numbers are indicated by letters which are not operated upon, but in addition, in order to facilitate the task of following and comprehending the argument, specific numerical values also appear in the diagram. These numbers are never referred to in the proof, but they tacitly accompany the reasoning and the reader can surely rely on them when trying to make sense of the argument. Here b is said to be the same parts of a as d is of c, and the claim is that b and d taken together is the same part of a and c taken together that b is of a.

In this proof, Campanus added yet another original idea not previously found in this context, at least not as explicitly as here: he associated to the idea of "being parts of" the idea of a pair of numbers that represent the relation in a concrete way, a numerator and a denominator respectively. Thus when we say for example "three fifths", the three measures (*numerat*) and the five denominates (*denominat*). Accordingly, in the figure, when we say that b is parts of a, we mean that these parts are numbered by a number h and denominated by another number k. Similar is the case for d being parts of c. In the diagram we also find the numbers e and f. These are the "parts" that arise in the argument, e in relation with b and f in relation with d. "By hypothesis", says Campanus, e is a part of b denominated by h and e is a part of a denominated by k. Likewise, f is a part of d denominated by h and f is a part of c denominated by k.

Notice that in the diagram accompanying Euclid's proof, *AG*, *GB* represent parts of C that taken together make the number *AB*. Likewise, *DH*, *HE* represent parts of F that taken together make the number *DE*. As many parts of C as taken together make *AB* so many parts of F taken together make *DE*. Hence, one can apply VII.5 and conclude that "whatever part *AG* is of C, the same part also is the sum of *AG*, *DH* of the sum of C, F." The same is then concluded for *GB* vis-à-vis C, and for the sum *GB*, *HE* vis-à-vis the sum of C, F. And finally, "whatever parts AB is of C, the same parts also is the sum of AB, DE of the sum of C, F."[15]

Now, the segments e, f play in Campanus' proof the same roles played in Euclid's proof by AG, DH respectively. But there is also an important difference: in Euclid there is no clue as to how C and F are divided into parts, whereas here it is said explicitly that a and c are divided k times each into e and f respectively. Now, in Euclid the *multiplicity* has a different status than the *magnitudes* about which the propositions are formulated and whose multiplicities we examine. The two are treated quite differently. This was particularly important in the case of Book V, where the multiplicities are of magnitudes in general. It also surfaces in the case of Book VII, given that in the diagrams (i.e., the diagram of VII.6) there is nothing parallel to what h and k are in Campanus' proof (i.e, the multiplicities). But here, as a consequence of Campanus' general approach in which every number referred to in the proof is indicated with a letter, and every such number is assigned a specific value in the diagram, the kind of separation consistently followed by Euclid has implicitly weakened, and both multiplicities and magnitudes are represented by the same kind of numbers!

It would make no sense, in terms of Campanus' approach, to refer to the multiplicity of the parts, and not to assign it a letter that helps refer to it and also a value that helps follow the diagram. As a consequence, in Campanus' treatment, the multiplicities and the numbers multiplied become much closer in nature than they were with Euclid. What was essentially a *technical* modification of a proof—motivated by considerations of didactical clarity—implicitly incorporated into the core of the mathematical theory thus developed, an important *conceptual* modification with implications well beyond the rather limited scope of the proof itself.

With this general background comments in mind, I shall just indicate now that the core of Campanus' argument for VII.6 lies in applying V.5 to e, f: their sum, g, turns out to be the same part of b and d taken together as b is of a. More specifically stated in the terms introduced by Campanus: g is a part of b and d taken together denominated by h, whereas g is a part of a and c taken together denominated by k. And this same relation applies to b and a respectively ("g is a part of b denominated by h, and g is a part of a denominated by k"). This is what was to be proved.

The last and very important point to mention about Campanus concerns a collection of fifteen commentaries added after the proof of proposition IX.16. These commentaries comprise, among other things, arithmetic versions of propositions from Book II that embody interesting distributivity-like properties. I already mentioned above al-Nayrīzī's AN-IX.16, which was a generalized version of Euclid's IX.15 and to which al-Nayrīzī appended as comments his own arithmetic versions of II.1–II.4. Campanus also added here his own commentaries, and most likely he did so by following on al-Nayrīzī's footsteps (Busard, 2005, 38).

One may wonder if there is some mathematical reason why, when al-Nayrīzī decided to add his commentaries, he chose precisely this particular proposition to do so. One may likewise ask if Campanus followed suit for the same reason. A speculative, but seemingly natural answer may be given with reference to Euclid's own proof of IX.15 and a peculiar feature that appears in it, namely, that several stages of the argument rely on what can be seen as arithmetic versions of II.3 and II.4. I already mentioned Euclid's consistent adherence to the principle of separating realms throughout the *Elements*. It is thus remarkable, that precisely here we have an interesting case of transgressing that principle. Let us see some of the details.

I start by going back to Euclid's of IX.15 whose original formulation was already stated above. The figure that accompanies his proof, as known to us, is Figure 14. Here A, B, C are the three given numbers in continued proportion. Proposition VIII.2 warrants the existence of two numbers, DE, EF such that

[15] See Itard (1961, 93–97) for what he considers to be a problematic aspect of Euclid's proofs for VII.6 and VII.8. See Taisbak (1971, 42–48) for additional, but quite different kind of comments.

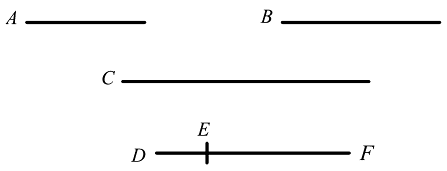

Figure 14. Euclid's *Elements* IX.15.

$A = DE^2$; $B = DE \cdot EF$; and $C = EF^2$. In addition, by VIII.22, DE, EF are mutually prime. Now, in the course of the proof, Euclid considers several *products* involving DF, DE, EF and their squares. Using various propositions of Book VII (22, 24, 25), he can establish certain relations of mutual primality among them. But in other places he also needs to consider cases where the numbers and their squares are *added* to one another, and it so happens that in Books VII and VIII there is only *one* proposition (VII.28) that handles cases of adding mutually prime numbers. It reads as follows:

> **VII.28:** If two numbers be prime to one another, the sum will also be prime to each other of them; and, if the sum of two numbers be prime to any one of them, then the original numbers will also be prime to one another.

Euclid indeed invokes this proposition in order to prove that DF is relatively prime to both DE and EF. But in the other stages in the proof he deals with additions not covered for VII.28 and that can be interpreted as arithmetic cases of the situation covered (for the geometric case) by II.3 and II.4. Thus, for instance:

- The product of FD, DE is the square on DE together with the product of DE, EF.
- The squares on DE, EF together with twice the product of DE, EF are equal to the square on DF.

Now, in preparation for this proof, Euclid could have conceivably formulated and proved the purely arithmetic propositions needed here, but for some reason he declined to do so and preferred to deviate from his self-imposed, strict separation of domains and use directly these arithmetic versions of his own geometric results.[16] As a matter of fact in its original, geometric version, II.3 is not applied anywhere in the entire treatise. It is only here that its basic idea is used in some proof, but then in this arithmetic version. Euclid did not add a comment or a word of warning about this peculiar transgression, and we can only speculate about the reason for this choice. But at the same time it would seem that if commentators such as Al-Nayrīzī or Campanus looked for a convenient place to connect arithmetic and geometric considerations, then this would be an ideal place to do so.

It is nevertheless worth mentioning that while Campanus' IX.16 reformulated (as also did al-Nayrīzī) Euclid's IX.15 in a more general way that refers to *any* multiplicity of numbers, in the proof he uses only four, exactly as Euclid did. On the other hand, his proof followed an argument that completely differs from Euclid's and that makes no use at all of II.3 or II.4.

A reader of Campanus who was also acquainted with Jordanus' *Arithmetica* (if there was any) would have easily recognized the close relationship (sometimes verbatim repetition) of Campanus' comments and Jordanus' basic rules of arithmetic discussed in the previous section. This is in line with Campanus' intention to turn the arithmetic books of the *Elements* into a foundational kind of text, along the lines of the *Arithmetica*. Book VII opened by rehearsing Jordanus' attempt to provide an axiomatic foundation, yet

16 See Mueller (1981, 108 ff.) for a broader discussion of Euclid's use of geometric arguments and analogies in arithmetical contexts. For example, Mueller indicates (p. 108) that in Book X, Euclid proves two *lemmata* while invoking an arithmetic analogue of II.6.

Campanus also used the opportunity to include those elementary propositions that Jordanus had developed in the first chapter of his book. As with Book VII, however, some of the technical changes that Campanus introduced in his presentation lead to some noteworthy differences. Let us consider some of them.

Campanus' first two comments state two symmetric, distributive-like rules involving products and additions. Their enunciations are parallel to, but not identical with, Jordanus' A-I.9 and A-I.10. This is how they appear in the text:

> **C-IX.9-1:** That which is made by multiplying a number by as many as we wish equals that which is made by multiplying it by them. [i.e., by the said many numbers]

> **C-IX.9-2:** That which is made of as many numbers as you wish in one, equals that which is made by their sum on it.

If we were to render these two propositions symbolically, according to what Campanus actually does in the specifications and in the proofs, we obtain the following expressions, which are indeed parallel to Jordanus':

***C-IX.9-1*:** $a \cdot b + a \cdot c + a \cdot d + \ldots = a \cdot (b + c + d + \ldots)$.
***C-IX.9-2*:** $b \cdot a + c \cdot a + d \cdot a + \ldots = (b + c + d + \ldots) \cdot a$.

Now, Campanus' proof of C-IX.9-1 is based on directly applying Euclid's VII.5. In contrast, you may recall, Jordanus had relied on II.1 to prove the proposition in his treatise which corresponds to this one (i.e., A-I.9). In other words, these two medieval authors understood the need to prove the distributivity of the product over the sum of any number of summands as a purely arithmetic result, but while Jordanus arithmetized an idea that has appeared in Euclid in a geometric context (Book II), Campanus preferred to do so by arithmetizing what had appeared originally as a general statement about equimultiplicity of magnitudes in general. For the second result in this pair, C-IX.9-2, Campanus did not divert from Jordanus' way and relied, like him, on the commutativity of the product (i.e, C-VII.17, which is parallel to Euclid's VII.16).

Now, in spite of his different approach to proving C-IX.9-1, Campanus found it convenient to stress that "the first of the second" (i.e., Euclid's II.1) states the same thing as C-IX.9-1 but for lines. And he also added similar statements in the following commentaries, 4 to 12, with relation to each of the propositions II.2-II.10 respectively. The first three of these correspond to the distributivity-like properties II.2–II.4, and they are proved by direct application of the first two rules, C-IX.9-1 and C-IX.9-2.

The important point to notice concerning these commentaries is that for the sake of their proof, Jordanus had had to start with a distributivity-like property for *multiplicities* (A-I.6). This provided the basis for proving other statements for arithmetic that are truly parallel to those of geometry, in the sense that they refer to a multiplication of *number by numbers*. Campanus, in turn, could base his proof directly on VII.5, which already handled the distributivity-like property of the multiplicities.

Campanus' version of the *Elements* had a decisive influence on the way that the treatise was read and understood over the following generations, particularly in relation with the issue of the relationship between arithmetic and geometry (Corry, 2013, 692). This is of course also the case concerning distributivity-like properties. Of particular importance in this regard is the addition of purely arithmetic versions to Book IX. Readers of the Campanus version, or of any other work derived from it, would now have good grounds—and by all means better grounds than those of a reader of any previous treatise—for seeing these properties as inherently arising within the purely arithmetic realm, without any need for additional support coming from geometric considerations.

6. Concluding remarks

The above account of the ways in which distributivity-like properties are formulated and proved in the works of Jordanus Nemorarius and Campanus de Novara affords an interesting, and I believe quite original, perspective on the medieval conceptions of arithmetic and the question of its foundations. Aware of the absence of a systematic foundation in the arithmetical books of the *Elements*, parallel to that of the geometrical ones, both authors devoted considerable, focused efforts to present what in their views should count as the basic propositions of this type of knowledge. Contemporary conceptions about the relationship between discrete and continuous magnitudes and the legitimate ways to handle them had become much more flexible than those typical of the *Elements* in its original formulations, and the distributive-like properties that Jordanus and Campanus discussed in their treatises bear witness to the more flexible approaches to the idea of number that had started to develop by then.

The next historical stage in considering the history of presentations of distributivity-like results within the Euclidean tradition was already within the world of the printed text, which was inaugurated with the 1482 Ratdolt edition of the *Elements* and what later on consolidated with the Commandino edition of 1572. These printed versions differed in important senses from previous, medieval ones, among other things because of the strong influence of the Campanus version, which was among the latest to be produced in the pre-print era. Hence the importance of the topics discussed in this article, as they shed light on the way in which the renaissance authors read their Euclid, and on the way that the relationship between arithmetic and geometry (and between discrete and continuous magnitudes) was to be conceived.

The changing relationships between geometry and arithmetic and the vigorous trends of symbolic algebraic techniques that began to attract increased attention opened the way to additional perspectives on the treatment of more general ideas of distributivity and on their place in the overall economy of mathematical knowledge. These new perspectives are worth further, detailed attention but of course they cannot be pursued here and I leave them for a future opportunity.

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