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The splitting data of cohomology classes

By

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I. Introduction. Let G be a finite group, M a G-module and assume given $\alpha \in \hat{H}^*(G, M)$, where $\hat{H}(G, -)$ denotes Tate cohomology, see Brown's book [1] ch. VI.

Denote by $\mathrm{Res}_H^G M$ the H-module obtained by *restriction* of scalars from G to H, where H is any subgroup of G. Then $\mathrm{res}_H^G\colon \hat{H}^*(G, M) \to \hat{H}^*(H, \mathrm{Res}_H^G M)$ is the homomorphism induced on the cohomology groups.

The *splitting data* of α is defined to be the set of subgroups of G to which the restriction of α is zero; we denote it by $D(\alpha)$:

$$D(\alpha) = \{H \leqq G\colon \mathrm{res}_H^G(\alpha) = 0\}.$$

The basic characteristic of res is its transitivity:

$$\mathrm{res}_K^G = \mathrm{res}_K^H \circ \mathrm{res}_H^G,$$

for subgroups $K \leqq H \leqq G$. This proves that $D(\alpha)$ is "closed" with respect to subgroups, i.e.

$$H \in D(\alpha) \quad \text{and} \quad K \leqq H \Rightarrow K \in D(\alpha).$$

Another well-known property of res is its invariance with respect to inner automorphisms; by this we mean the following: let $H \leqq G$, $g \in G$ and denote $g H g^{-1}$ by ${}^gH$. There is an isomorphism ("change of groups")

$$c(g)\colon \hat{H}^*({}^gH, M) \to \hat{H}^*(H, M), \qquad \text{see [1], p. 80}$$

which commutes with restrictions; thus

$$c(g) \cdot \mathrm{res}_H^G(\alpha) = \mathrm{res}_{{}^gH}^G(\alpha) \cdot c(g), \qquad \text{see [1], p. 83}$$

which implies that

$$\mathrm{res}_H^G(\alpha) = 0 \Leftrightarrow \mathrm{res}_{{}^gH}^G(\alpha) = 0.$$

In other words, $D(\ )$ is "closed" with respect to conjugation, i.e.,

$$H \in D(\alpha) \Leftrightarrow {}^gH \in D(\alpha).$$

Finally, let $H \leqq G$ and let $H_1, \ldots, H_m$ be a complete list of non-conjugate Sylow subgroups of H (i.e. corresponding to the different primes $p_1, \ldots, p_m$ dividing $|H|$). The injectivity of $\mathrm{res}^H_{H_i}$ on the p_i-primary part of $\hat{H}^*(H, M)$ implies:

$$\mathrm{res}^G_H(\alpha) = 0 \Leftrightarrow \mathrm{res}^G_{H_i}(\alpha) = 0 \quad \text{for all} \quad i = 1, \ldots, m.$$

This implies part (iii) of the following.

Proposition 1. *The set $D(\alpha)$ has the following properties*

(i) *If $H \in D(\alpha)$ and $K \leqq H$ then $K \in D(\alpha)$.*
(ii) *If $H \in D(\alpha)$ and $g \in G$ then ${}^gH \in D(\alpha)$.*
(iii) *$H \in D(\alpha)$ if and only if all Sylow subgroups of H belong to $D(\alpha)$.*

Question. Given a set of subgroups D satisfying (i), (ii), (iii), is it realizable as $D(\alpha)$ for some $\alpha \in \hat{H}^k(G, \mathrm{M})$; i.e. can a *solution pair* (M, α) be constructed for any k and D?

It will be shown below that this is possible. Moreover, we can even give restrictions on the *order* of α. To explain this we define the *index* of D, denoted by $(G:D)$:

$$(G:D) = \text{g.c.d.}\,\{(G:H): H \in D\}.$$

Suppose now that $D = D(\alpha)$ for some $\alpha \in \hat{H}^*(G, M)$, and that $p \mid (G:D)$.

Let P be a maximal p-group belonging to D, and let Q be a p-Sylow subgroup of G containing P. Since $p \mid (G:D)$, $P \neq Q$. This means $Q \notin D$. Thus $\beta = \mathrm{res}^G_Q(\alpha) \neq 0$. However, β is of order a power of p, which implies

$$p \mid \mathrm{order}(\alpha).$$

Thus we see: if p divides $(G:D)$ and D is to be equal to $D(\alpha)$ then p must divide the order of α. We will see that this condition is also sufficient.

Theorem. *Let D be a set of subgroups satisfying* (i), (ii), (iii) *of Proposition* 1, *and let m be an integer dividing $(G:D)$ and such that every prime diving $(G:D)$ divides also m. Then D is realizable as $D(\alpha)$ for some $\alpha \in \hat{H}^k(G, M)$, with α of order m. Moreover M can be taken to be a finitely generated $\mathbb{Z}G$-Module which is torsion-free.*

We will say that (M, α) solves the problem (G, D, m). There are related questions we do not touch such as finding a "smallest" M or at least an M for which $\hat{H}^k(G, M)$ is smallest. We briefly mention our motivation for asking the question above. If $k = 2$, then $\alpha \in \hat{H}^2(G, M)$ which "classifies" the group extensions

$$0 \to M \to \Gamma \to G \to 1.$$

The extension corresponding to α splits precisely over the subgroups in $D(\alpha)$. Since we assumed M is torsion-free this implies that $D(\alpha)$ is also a complete list of finite subgroups of Γ, up to some equivalence relation weaker than isomorphism (i.e. it implies isomorphism). Such groups Γ are useful to have in that they give interesting examples for problems related to "ranks" of crossed products (see [2], [3]). In another paper planned by one of us the main conjecture concerning "Euler-Goldie" ranks will be proved.

We also want to acknowledge a debt to A. Mann for bringing to our attention an error.

II. Preliminaries. Given a finite group G of order n one knows $\hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/(n)$. We use dimension shifting to obtain G modules for which $\hat{H}^k$ is $\mathbb{Z}/(n)$.

Lemma 2. *Let G be a finite group of order n then for any $k \in \mathbb{Z}$ there is a G module N which as an abelian group is finitely generated and torsion-free, and $\hat{H}^k(G, N) = \mathbb{Z}/(n)$.*

Proof. Since G is finite every module is a submodule and a quotient of an induced module. Thus, the existence of such a module N follows if we know one for $k+1$ or for $k-1$. Thus, starting from $k = 0$ we can go up and down, which implies the result for all k. An alternative, rather more complicated, way to prove the lemma directly for $k = 2$ is described in [2].

We make heavy use of Shapiro's Lemma ([1], p. 136). If M is any H module, where $H \leqq G$, then there is a natural isomorphism

$$s_H^G\colon \hat{H}^*(H, \mathrm{M}) \approx \hat{H}^*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M).$$

$\mathbb{Z}G \otimes_{ZH} M$ will be denoted by $\mathrm{Ind}_H^G M$. If $\alpha \in \hat{H}^*(H, M)$, the corresponding element in $\hat{H}^*(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M)$ will be denoted by $\alpha' = s_H^G(\alpha)$.

The following lemma is important.

Lemma 3. *Let G be any finite group, with subgroups K, H such that $K \leqq H$. If M is an H-*Module, *$\alpha \in \hat{H}^*(H, M)$ and α' is the corresponding element in $\hat{H}^*(G, \mathrm{Ind}_H^G M)$ then*

$$\mathrm{res}_K^G(\alpha') = 0 \Leftrightarrow \mathrm{res}_K^H(\alpha) = 0.$$

To see this note that there is an H module isomorphism

$$\mathrm{Res}_H^G \cdot \mathrm{Ind}_H^G M \approx M \oplus M'$$

such that, if π denotes the projection of the right hand side to M then

$$\pi_*(\mathrm{res}_H^G \cdot s_H^G(\alpha)) = \alpha.$$

This is easily checked directly on the resolution. The lemma clearly follows.

III. Proof when G is a p-group. This is the main part of the proof. We assume $|G| = p^s$ and work by induction on s.

The case $s = 0$ is trivial while $s = 1$ is very simple: we can assume G is not in D (or else the problem is trivial) so if $|G| = p, D$ consists of $\{1\}$ only, and we can take the N given by Lemma 2 and α a generator of $\hat{H}^k(G, N) \approx \mathbb{Z}/(p)$. Let now $|G| = p^s$ with $s > 1$ and let H be a subgroup of index p. One knows H is normal and, if $K \leqq G, K \nleq H$ then $KH = G$.

Lemma 4. *Let $|G| = p^s$, H a subgroup of index p and $K \leqq G$. Let N be an H module and $\beta \in \hat{H}^k(H, N)$. Then*

$$\mathrm{res}_K^G \cdot s_H^G(\beta) = 0 \Leftrightarrow s_{K \cap H}^K \cdot \mathrm{res}_{K \cap H}^H(\beta) = 0.$$

Proof. If $K \subset H$, i.e. $K \cap H = K$ then $s_K^K = \mathrm{id}$ and this is Lemma 3. If $K \not\subset H$, i.e. $KH = G$ then by a special case of the Mackey theorem (see [1], p. 69, prop. (5.6.b))

$$\mathrm{Res}_K^G \cdot \mathrm{Ind}_H^G = \mathrm{Ind}_{K \cap H}^K \cdot \mathrm{Res}_{K \cap H}^H \quad \text{(natural isomorphism of functors).}$$

This implies, in this case, the stronger fact:

$$\mathrm{res}_K^G \cdot s_H^G(\beta) = s_{K \cap H}^K \cdot \mathrm{res}_{K \cap H}^H(\beta). \qquad \square$$

Still assuming $(G: H) = p$ let

$$D_H = \{K \in D: K \leqq H\}.$$

We ask: what is the index of D_H in H? If there exists a subgroup K in D of minimal index (i.e. such that $(G: D) = (G: K)$) *not contained in* H then we see that since $G = HK$

$$(G: K) = (H: H \cap K)$$

and $(H: D_H) = (G: D)$. If, on the other hand, every $K \in D$ of minimal index is contained in H then clearly

$$p \cdot (H: D_H) = (G: D).$$

We can summarize this by

Lemma 5. *If* $|G| = p^s$, *and* $H < G$ *of index* p *then either* $(H: D_H) = (G: D)$ *or*

$$p \cdot (H: D_H) = (G: D).$$

We now prove the theorem. We distinguish two cases.

Case I. D contains *all* proper subgroups of G. Then $(G: D) = p$ and necessarily $m = p$. We get a solution pair (N, α) by taking N such that $\hat{H}^k(G, N) = \mathbb{Z}/(p^s)$, β a generator of $\hat{H}^k(G, N)$, and $\alpha = p^{s-1}\beta$.

Case II. Not all subgroups of index p lie in D. Let these (not in D) be $H_1, \ldots, H_r$ and denote

$$D_i = \{K \subseteqq H_i: K \in D\}.$$

We want to construct a G module M and $\alpha \in \hat{H}^k(G, M)$ of order $m = p^t$ with splitting data precisely D. For each $i = 1, 2, \ldots, r$ let

$$m_i = \min\{(H_i: D_i), m\}$$

and let (N_i, β_i) be a solution pair for (H_i, D_i, m_i), i.e. group H_i, splitting data D_i and order m_i. This exists by the inductive assumption. Let

$$M_i = \mathrm{Ind}_{H_i}^G N_i,$$

$$\alpha_i = s_{H_i}^G(\beta_i).$$

If $M = \bigoplus_{i=1}^r M_i$ with natural inclusions $\varphi_i: M_i \to M$ let

$$\alpha = \varphi_{1*}\alpha_1 + \cdots + \varphi_{r*}\alpha_r.$$

It is easily checked that

$$D(\alpha) = \bigcap_{i=1}^{r} D(\alpha_i).$$

Now suppose $K \in D$. If $K \subseteqq H_i$ then $K \in D_i$ so $K \in D(\alpha_i)$ by lemma 3 (or 4). If $K \not\subset H_i$ then $K \cap H_i \in D_i$ so $K \in D(\alpha_i)$ by lemma 4. Thus, $D \subseteqq D(\alpha)$. If $K \notin D$, $K \neq G$, let H be a subgroup of index p containing K. $H \notin D$ since $K \notin D$; say $H = H_1$. Thus $K \notin D$; so $K \notin D(\alpha_1)$, by lemma 3. This shows that (M, α) is a solution pair in that

$$D(\alpha) = D.$$

We must still take care about the *order* of α. Clearly

$$\text{order}(\alpha_i) = \text{order}(\beta_i) = m_i$$

so order $(\alpha) = \max\{m_i\colon 1 \leqq i \leqq r\}$. The possibilities for this number are m or $p^{-1}m$. In the first case we are done while in the second case we modify (M, α) as follows. Let (N, β) be such that $\hat{H}^k(G, N) = \mathbb{Z}/(p^s)$ and β a generator. Recall that $m = p^t \leqq p^s$. Let $M' = M \oplus N$ with

$$\varphi\colon M \to M', \quad \psi\colon N \to M'$$

the "natural" inclusions and $\alpha' = \varphi_*(\alpha) + p^{s-t}\psi_*(\beta)$. Clearly order$(\alpha') = m$, while $D(\alpha') = D(\alpha) \cap D(p^{s-t}\beta) = D(\alpha)$ since $D(\alpha) \subset D(p^{s-t}\beta)$ = set of *all* subgroups of order dividing p^{s-t}. This completes the construction when G is a p-group.

Remark. We could follow different routes and construct first α of order p (say) and $D(\alpha) = D$, then modifying it, as above, to get α' of order m and same splitting set.

IV. The general case. Suppose now that $(G\colon D) = p_1^{r_1} \cdots p_n^{r_n}$, where D is as in Proposition 1 and $p_1, \ldots, p_n$ are distinct primes, $r_i > 0$. Note that $|G|$ might possibly have more primes dividing it, but for us the p_i are the relevant ones. Let $m = p_1^{e_1} \cdots p_n^{e_n}$ where $1 \leqq e_i \leqq r_i$ for $1 \leqq i \leqq n$. We must construct a G module M and $\alpha \in \hat{H}^k(G, M)$ of order m and $D(\alpha) = D$. For each $i = 1, \ldots, r$ let H_i be a p_i-Sylow subgroup of G and let

$$D_i = \{K < H_i\colon K \in D\}.$$

It is easily seen, using assumption (iii), that

$$(H_i\colon D_i) = p_i^{e_i}.$$

Let (N_i, β_i) be a solution pair for the "problem" $(H_i, D_i, p_i^{e_i})$, i.e., $\beta_i \in \hat{H}^k(H_i, N_i)$, order $(\beta_i) = p_i^{e_i}$ and $D(\beta_i) = D_i$. The existence for this problem is known from the former section. Let

$$M_i = \operatorname{Ind}_{H_i}^G N_i, \quad M = M_1 \oplus \cdots \oplus M_n$$

$$\alpha_i = s_{H_i}^G(\beta_i), \quad \alpha = \varphi_{1*}\alpha_1 + \cdots + \varphi_{n*}\alpha_n$$

where $\varphi_i\colon M_i \to M$ the inclusion.

It is clear that α has order m and it remains to show $D(\alpha) = D$. Let $K \in D$. To show $\operatorname{res}_K^G(\alpha) = 0$ it suffices to show

$$\operatorname{res}_K^G(\varphi_{i*}\alpha_i) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

To simplify the notation let us drop the φ_{i*} in the computation below, and write α_i for $\varphi_{i*}\alpha_i$. Since α_i has order $p_i^{e_i}$ it suffices to check

$$\operatorname{res}_{K_i}^K \operatorname{res}_K^G(\alpha_i) = \operatorname{res}_{K_i}^G(\alpha_i) = 0$$

where K_i is a p_i-Sylow subgroup of K. But by lemma 3 this is indeed the case. This shows $D \subseteqq D(\alpha)$.

Let $K \notin D$. Then for some $1 \leqq i \leqq n$ the p_i-Sylow subgroup of K, K_i, is not in D_i (or else, by (iii), $K \in D$). Another application of Lemma 3 shows that $\operatorname{res}_{K_i}^G(\alpha_i) \neq 0$. This completes the proof.

References

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