Heinrich Weber: *Lehrbuch der Algebra* (1895-96)

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First edition in 2 vols., Braunschweig, F. Vieweg und Sohn, 1895-96, First edition of Vol. III: *Elliptische Funktionen und algebraische Zahlen* published separately in 1891 (xvi, 733 p.).

2d. Edition, Vol. I-III: Braunschweig, F. Vieweg und Sohn, 1898-1908.

Reduced ed. in one vol., Braunschweig, F. Vieweg & sohn, 1912. x, 528 p.

Reprint, New York, Chelsea Pub. Co., 1961.

French Translation of the 2d. ed. (by J. Griess): *Traité d'algebre supérieure*, Paris : Gauthier-Villars, 1898.

As the last important textbook on algebra published in the nineteenth century Weber's *Lehrbuch* presents a faithful image of algebraic knowledge and of the way in which the discipline was conceived at the time. Although many of the abstract concepts that became central to the structural conception of algebra after 1930 were well known to Weber, they play a relatively secondary role in his presentation of the discipline. Algebra still appears here as the discipline of polynomial equations and polynomial forms.

1. Background

The discipline of algebra underwent significant changes between the last third of the nineteenth century and the first third of the twentieth. These changes comprised the addition of important new results, new concepts and new techniques, as well as meaningful changes in the way that the very aims and the scope of the discipline were conceived by its practitioners. Over the century, algebraic research had meant mainly research on the theory of polynomial equations and the theory of polynomial forms, including algebraic invariants. The ideas implied by Evariste Galois's works became increasingly visible and central since the latter's publication by Joseph Liuoville in 1846. Together with important progress in the theory of fields of algebraic numbers, especially in the hands of Leopold Kronecker and Richard Dedekind, they gave rise to an increased interest in new concepts such as groups, fields and modules.

A very popular textbook of Algebra since the middle of the century was the *Cours d'algèbre supérieure* by Joseph Serret that underwent three editions in 1849, 1854 and 1866, respectively. In these successive editions it gradually incorporated the techniques introduced by Galois and it became the first university textbook to publish a full exposition of the theory. Still, it continued to formulate the main results of Galois theory in the traditional language of solvability dating back form the works of Joseph-Louis Lagrange and Niels Henrik Abel at the beginning of the century and in doing so, it did not even include a separate discussion of the concept of group. A second contemporary textbook was Camille Jordan's *Traité des substitutions et des equations algèbriques* (1870) which already presented a more elaborate presentation of the theory of groups, but still treated this theory as subsidiary to the main task of discussing solvability conditions for polynomials.

A completely different image of algebra is the one embodied much later in Bartel L. van der Waerden's textbook of 1930 *Moderne Algebra.* Here we are presented for the first time with a discipline at the center of which stands the general idea of an abstract algebraic structure that is instantiated in various particular species such as groups, rings and fields calling for being elucidated with the help of a standard set of tools. This is the image that came to dominate research in the twentieth century.

Weber's *Lehrbuch der Algebra* stands midway between these two poles. It incorporates an entire body of new individual ideas and techniques developed along the nineteenth century, and in doing so it provides a full picture of what algebraic knowledge looked like at the time. In spite of the knowledge it adds over books like Serret's or Jordan's, the picture of algebra it presents does not differ essentially from theirs. On the other hand, in spite of including a great deal of material that would eventually be incorporated as at the basis of van der Waerden's presentation, it does not envisaged the kind of fundamental change in conception that *Moderne Algebra* intended to imply.

2. Heinrich Weber's Career

Heinrich Weber (1842-1913) studied in Heidelberg and Leipzig. He habilitated in Königsberg in 1866, and taught there until 1883, except for the years 1870 to 1875, when he was professor at the ETH Zürich. He later spent several years at the Charlotenburg technological institute (Berlin), and in Marburg, and was Ordinarius in Göttingen between 1892 and 1895. Finally he moved to Strasbourg, where he remained until his death in 1913. Weber's Königsberg years were the most productive of his successful career. The tradition of analysis and mathematical physics developed in this university, under the leadership of Carl Gustav Jacobi, Franz Neumann and, somewhat later, Friedrich Richelot, was a main force behind the increasing dominance attained by Germany in the mathematical world over the nineteenth century. Weber was but one of the outstanding mathematicians whose names came to be connected with that school. The early careers of Adolf Hurwitz, Hermann Minkowski and David Hilbert were also later associated with this institution and their works were decisively influenced by its tradition. Weber's mathematical activities spanned many different domains of mathematics such as algebra, number theory and mathematical physics. In his historical account of the development of mathematics in the nineteenth century, Felix Klein described Weber as the most versatile representative of that trend, that Klein himself so proudly felt part of and that sought to elaborate the interconnections between mathematical domains such as the theory of invariants, the theory of polynomials equations, the theory of functions, geometry and the theory of numbers (Klein 1926-27, Vol., 275).

Weber's *Lehrbuch* was only one among several important works he published and that reached a wide mathematical audience. Thus, for instance, together with J. Wellstein, he published the widely-read *Enzyklopädie der Elementarmathematik*. Weber also collaborated with Dedekind in two additional, important projects. One was a seminal article on the theory of algebraic functions, published in 1882. The second one was the edition of Bernhard Riemann's mathematical papers. It is very likely that without Weber's active help, Dedekind would have never completed the task that he undertook when he agreed to produce that edition.

When the first edition of Volume I of the *Lehrbuch der Algebra* appeared in 1895, Heinrich Weber was well-aware of the latest advances in algebra, and in particular of the possibility of formulating new algebraic concepts in purely abstract terms. As a matter of fact, in 1893 he had been the first to provide abstract definitions of both groups and fields within the framework of a single article. Moreover, his research on algebraic functions in collaboration with Dedekind shows that Weber was deeply acquainted with the latter's theory of ideals, a theory that played a central role in the rise of the structural approach to algebra. And yet, when the time came for presenting the current state of knowledge in the discipline, he chose to present such concepts as playing only a relatively marginal role within it.

3. The Introduction to the *Lerhbuch*

In the preface to the first volume of the *Lehrbuch* Weber explained that the development of algebra over the preceding decades had rendered the existing textbooks obsolete and had brought about the need for a new coherent presentation of results and their applications. Among the books he had in mind were Serret's *Course* and Jordan's *Traité*. The aim of his first volume was to present the "elementary parts of algebra", namely all what may be subsumed under the designation of "formal algebraic manipulation" (*Buchstabenrechnung*), beginning with the rules for the determination of the roots of an equation and finishing with an exposition of Galois theory. Weber explicitly acknowledged Dedekind's influence in consolidating his long-standing interest in algebra. This influence acted mainly through the notes of Dedekind's 1857-58 Göttingen lectures on Galois theory, the manuscript of which Weber had had the opportunity to read.

The problem of finding the roots of polynomial equations dominates a considerable portion of the book. Like all previous books in algebra, the whole theory of polynomials appears here as conceptually dependent on a thorough knowledge of the properties of the various systems of numbers. The fashion in which these systems are introduced in order to provide the necessary conceptual infrastructure differs considerably, however, from previous ones in that it is strongly based on the notion of set (*Mannigfaltigkeit oder Menge*). Weber introduced the concept —following Dedekind—in what we would call today a naive formulation: a system of objects or elements of any kind, such that for any given object one can always say whether it belongs to the set or not. Weber also introduced additional, related concepts such as ordered sets, discrete and dense (*dicht*) sets (exemplified by the integers and the rationals), cuts (*Schnitte*) and continuity (*Stetigkeit*)—all of them as previously defined by Dedekind.

Within this framework of ideas, the rational numbers are introduced as a dense, but discontinuous set and the real numbers as the set of cuts of the rationals. In spite of its markedly abstract orientation, Weber's definitions of the various number systems essentially differs from what became the standard in twentieth-century mathematics. He conceived these systems as well-known, specific mathematical entities whose properties—although originating in free acts of creation of the human spirit—are given once and for all in advance. In the image of algebra embodied in Weber's book, the algebraic properties of number systems do not derive from those of some more basic, underlying, abstract algebraic structures. Rather it is the other way around: algebra is based on the given properties of the number systems.

The introduction of the *Lehrbuch* closes with a remark on the formal manipulation of symbols. One can distinguish two main forms of the latter: identities and equations. Algebra, wrote Weber, is the discipline whose aim is the resolution of equations. This statement is not mere lip-service to the prevailing views. The contents of the book faithfully reflected this declared central role of equations, whereas other issues, such as the study of groups, appear as conceptually subsidiary to this aim. In fact, besides groups, no other abstract algebraic concept (fields, modules, rings, etc.) is systematically investigated in the *Lehrbuch*.

4. The Three Volumes

The first volume of the *Lehrbuch* comprises three books. The first two of them deal with the classical theories of polynomial equations. Within this framework, Chapters III and IV provide interesting evidence of the attachment of Weber to nineteenth century images of algebra. Thus, for instance, in Chapter III, the concept of root of an equation is discussed in terms that may be classified as "analytic": limits, continuity, $\varepsilon-\delta$ arguments etc. Arguments of this kind would later be excluded from standard, structural presentations of algebra. Likewise, Chapter IV deals with "symmetric functions," that had been used by Lagrange in his early research on solvability of polynomial equations. Later, the gradual development of Galois theory as the main tool for studying solvability of polynomial equations eventually rendered symmetric functions a rather dispensable tool, yet Weber included a treatment of them in the *Lehrbuch* as part of a tradition of which his approach to algebra was part and parcel. Under the conception of algebra characteristic of this tradition, a treatment of the theory of polynomial equations should include every particular technique devised to deal with their solvability, as he indeed did here.

In Book II one finds additional discussions that are analytic in character, and that would be excluded from later textbooks of algebra. This is the case, for instance, of the theorem of Sturm discussed in Chapter VIII. Sturm's problem concerns the question of how many real roots of a given polynomial equation lie between two given real numbers. This, and further similar problems, are solved with the help of derivatives and other analytical tools. Likewise, Weber discussed well-known approximation techniques: interpolation and Newton's method are mentioned among others in Chapter X. Chapter XI deals with roots of unity: no mention whatsoever, however, is made of their group-theoretical properties.

Galois theory is finally introduced in Book III, after nearly five hundred pages of discussion on the resolution of polynomial equations. First, a field of numbers is defined as a set of numbers closed under the four operations. Indeed, the concept is extended to fields of functions or to any set closed under the four operations of addition, multiplication, subtraction and non-zero division (pp. 491-492). However, although Weber referred here to his own article of 1893, in which he had insisted upon the potential interest involved in studying finite fields, in the *Lehrbuch*, he considered only (infinite) fields of characteristic zero. In no way did he research fields as an autonomous concept with intrinsic interest, even at the relatively elementary level that he did for groups.

Groups are mentioned for the first time as late as on page 511. But even here one does not find a general treatment of groups; this is left for later chapters. At this stage, Weber considered substitutions of one root of a function with another, substitutions that may themselves be composed to form a group, and, more specifically, a finite group (p. 513). Weber defined here a group of permutations—a concept which he used in the next chapter—and the Galois group of a given field.

Chapter XIV shows the application of groups of permutations to the theory of equations. First, Weber showed that any permutation may be decomposed into transpositions and cycles. He then defined some additional, basic concepts: subgroups (*Teiler*), and the cosets (*Nebengruppen*) determined by a given permutation, as well as the index of a subgroup of permutations. He explicitly stated that the aim of this whole section was to improve our understanding of the issues dealt with in the preceding chapter (p. 529). Thus the focus of interest does not lie in the study of the properties of the group of permutations as such, but only insofar as it sheds light on the theory of equations.

In the following chapters, Weber analyzed particular cases of equations using the insights provided by the already developed theory. The exposition culminates towards the end of the book, in Chapter XVII, where the algebraic solution of equations is systematically discussed. Weber acknowledged the centrality of this problem for the contemporary development of algebra, and the important contribution of group theory to its better understanding. Thus, he wrote (p. 644):

> One of the oldest questions which the new algebra has preferentially addressed is that of the so-called algebraic solution of equations, meaning the representation of the solution of an equation through a series of radicals, or their calculation through a series of root-extractions. The theory of groups sheds much light on this question.

It is in this section that Weber proves that the alternating group is simple— a result needed for the proof of the impossibility of solving the general fifth degree equation in radicals (pp. 649-652).

A thoroughly abstract definition of group, similar to that of Weber's own 1893 article appears only in the second volume of the *Lehrbuch*. After the basic concepts of the theory of groups were introduced in the first four chapters of the second volume in a general and abstract way, Weber stated the object of the abstract study of groups. His formulation stresses the need he felt to explain to contemporaries the meaning of the very use of abstract concepts of this kind (p. 121):

> The general definition of group leaves much in darkness concerning the nature of the concept.... The definition of group contains more than appears at first sight, and the number of possible groups that can be defined given the number of their elements is quite limited. The general laws concerning this question are barely known, and thus every new special group, in particular of a reduced number of elements, offers much interest and invites detailed research.

Weber also pointed out that the determination of all the possible groups for a given number of elements was still an open question. It had been recently addressed by Arthur Cayley, but only for the lowest orders.

In the following chapters, Weber discussed special instances of groups (groups of characters, groups of linear substitutions, polyhedric groups) and presented some applications of group theory, such as Galois theory, invariants, and others. The last part of the second volume deals with algebraic number theory. Following Dedekind, the fields considered are only fields of numbers, rather than abstract ones.

The third volume of the *Lehrbuch* appeared in 1908 and it embodies, in fact, a second edition of Weber's book on elliptic functions and algebraic numbers, first published in 1891. It deals with the reciprocal interrelations between problems and techniques of the theory of fields of algebraic numbers and of the theory of elliptic functions. It is important to notice that a complete description of contemporary images of algebra cannot fail to stress the importance of the connection established in this third volume between these two domains, algebra and the theory of elliptic functions. The kinds of conceptual and technical interconnections that were pursued by mathematicians like Weber during the second half of the nineteenth century in relation to algebraic problems cover a much broader spectrum than what the later, structural image of algebra may lead us to assume. The third volume of Weber's book touches upon some important portions of that spectrum. The existence of these kinds of interconnections underscores the difficulties inherent to the use of one and the same term (algebra) to denote the disciplines known by this name in the nineteenth and in the twentieth-century.

5. Impact

If one considers together the ideas appearing in Weber's 1893 and in his *Lehrbuch* one notices a complex picture of what algebraic knowledge was for him. This picture comprises elements of both classical nineteenth-century, as well as more modern, conceptions. The central issue of the first volume of the *Lehrbuch* was the resolution of polynomial equations, and its presentation remains similar to those of those appearing in earlier textbooks of algebra. All the concepts and techniques related to Galois theory (in particular the concepts of group and field) are introduced, to a large extent, only as ancillary to that central issue. By the end of the century, group theory was the paradigm of an abstractly developed theory, if there was any. Research on groups had increasingly focused on questions that we recognize today as structural, and, at the same time, the possibility of defining the concept abstractly had been increasingly acknowledged. More importantly, the idea that two isomorphic groups are in essence one and the same mathematical construct had been increasingly adopted. Weber's 1893 article exemplifies clearly this trend. Yet, in Weber's book, group theory plays a role which, at most, may be described as ambiguous regarding the overall picture of algebra. For, although in its second volume, the theory of groups is indeed presented as a mathematical domain of intrinsic interest for research and many techniques and problems are presented in an up-to-date, structurally-oriented fashion, the theory appears in the first volume as no more than a tool of the theory of equations (albeit, it is now clear, a central one). Weber's textbook, and much more so his 1893 article, bring to the fore the interplay between groups and fields abstractly considered more than any former, similar work. However, in spite of this, the classical conceptual hierarchy that viewed algebra as based on the essential properties of the number systems is not called into question in any of these two works.

Weber's *Lehrbuch* became the standard German textbook on algebra and underwent several reprints. Its influence can be easily detected, among others, through the widespread adoption of a large portion of the terminology introduced in it. This is not to say that all the terminology he used was widely adopted. Thus for instance we find in his book the term "metacyclic" groups (Vol. 1, p. 646), which denoted the group of an equation that can be fully solvable by radicals, or *Ordnung* (following Dedekind) to denote a ring of algebraic numbers.

At any rate, the image of algebra conveyed by Weber's book was to dominate the algebraic scene for almost thirty years, until van der Waerden's introduction of the new, structural image of algebra. But obviously, influential as the latter was on the further development of algebra, it did not immediately obliterate Weber's influence, which can still be traced to around 1930 and perhaps even beyond. One can notice this by looking at several books published in the 1920s, such as Leonard Eugene Dickson's *Modern Algebraic Theories* (1926) and Helmut Hasse's *Höhere Algebra* (1926). But the clearest sign of the *Lehrbuch* longstanding influence on algebraic activity, especially within Germany, is provided by the publication in 1924 of another textbook by Robert Fricke. Fricke wrote his book upon request of Weber's publisher in Braunschweig, F. Vieweg, after the *Lehrbuch* had sold out. In spite of the relatively long time since the original publication, and the many important adavnces in algebraic research since then, Fricke chose to essentially abide by the conception of algebra embodied in Weber's presentation He stressed this very clearly in the name he chose for his own textbook: *Lehrbuch der Algebra - verfasst mit Benutzung vom Heinrich Webers gleichnamigem Buche*.

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