History of Algebra

The term algebra usually denotes various kinds of mathematical ideas and techniques, more or less directly associated with formal manipulation of abstract symbols and/or with finding the solutions of an equation. The notion that in mathematics there is such a separate sub-discipline, as well as the very use of the term "algebra" to denote it, are themselves the outcome of historical evolution of ideas. The ideas to be discussed in this article are sometimes put under the same heading due to historical circumstances no less than to any "essential" mathematical reason.

Part I. The long way towards the idea of "equation"

Simple and natural as the notion of "equation" may appear now, it involves a great amount of mutually interacting, individual mathematical notions, each of which was the outcome of a long and intricate historical process. Not before the work of Viète, in the late sixteenth century, do we actually find a fully consolidated idea of an equation in the sense of a *single* mathematical entity comprising *two side*s on which operations can be simultaneously performed. By performing such operations the equation itself remains unchanged, but we are led to discovering the value of the unknown quantities appearing in it.

Three main threads in the process leading to this consolidation deserve special attention here:

(1) attempts to deal with problems devoted to finding the values of one or more unknown quantities. In Part I, the word "equation" is used in this context as a short-hand to denote all such problems, even though the point to be stressed is precisely the *absence* of the full idea of an equation

(1) the evolution of the notion of number, gradually leading to an elaborate conception of arithmetic, general and flexible enough to bear the algebra on it, and the concomitant increase in the willingness to accept the legitimate character of ever broader domains of numbers (rational, irrational, negative, complex)

(1) the gradual refinement of a symbolic language that favored the development of generalized algorithmic processes for solving problems

Babylonian and Egyptian Mathematics

Egyptian mathematical texts known to us date from about 1650 B.C. They attest for the ability to solve problems equivalent to a linear equation in one unknown. Later evidence, from about 300 B.C. indicates the ability to solve problems equivalent to a system of two equations in two unknown quantities, involving not only the quantity itself, but also its squares. Throughout this period there is no use of symbols; problems are stated and solved verbally, like in the following, typical example:

> Method of calculating a quantity,
> multiplied by 1 1/2 added 4 it has come to 10.
> What is the quantity that says it?
> Then you calculate the difference of this 10 to this 4. Then 6 results.
> Then you divide 1 by 1 1/2. Then 2/3 result.
> Then you calculate 2/3 of this 6. Then 4 results.
> Behold, it is 4, the quantity that said it.
> What has been found by you is correct.

Except for 2/3, for which a special symbol existed, the Egyptians expressed fractionary quantities using only “unit fractions”, i.e., fractions having 1 as nominator. For instance: ¾ would be written as a sum of one half and one quarter.

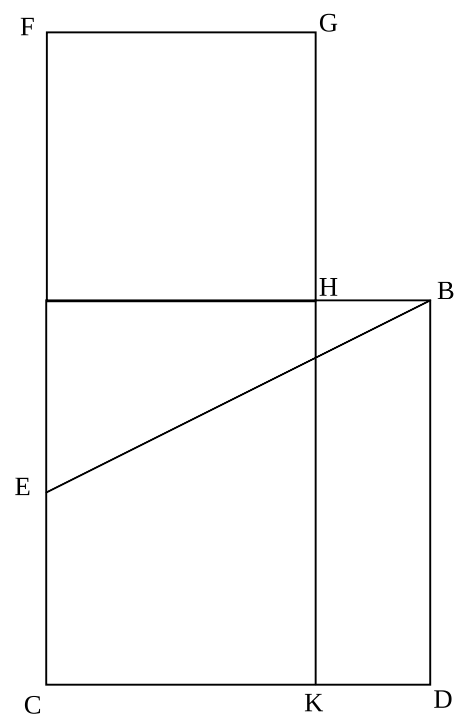
Babylonian mathematics dating from as early as 1800 B.C. has reached us by means of cuneiform texts preserved in clay tablets. Babylonian arithmetic was based on a well-elaborated, positional sexagesimal system (base 60). There is, however, no consistent use of zero. A great deal of Babylonian mathematics consists of tables: multiplication and reciprocal tables, squares, square and cube roots (though no cubes), exponentials and others.

Beside tables, there are problem texts involving the computation of an unknown number. These texts explain a procedure to be followed in order to find the number. This is illustrated by a specific example, rather than by abstractly describing its successive steps. The starting point could be relations involving specific numbers and the unknown, or its square, or systems of such relations. The number sought could be the square root of a given number, the weight of a stone, or the length of the side of a triangle. Many of the questions are phrased in terms of “concrete” situations (e.g.: partitioning a field among three pairs of brothers under certain constraints) yet their evidently artificial character makes it clear that such problems were meant as didactical, rather than practical, exercises.

Greek Mathematics: Proportion Theory, Elementary Arithmetic

A major milestone of Greek mathematics was the discovery by the Pythagoreans around 430 B.C. that certain ratios among pairs of magnitudes do not correspond to simple ratios among whole numbers. This surprising fact, which run counter to the most basic metaphysical beliefs of the Pythagoreans, became clear while investigating what appears to be the most elementary ratio between geometrical magnitudes, namely, the ratio between the side and the diagonal of a square (nowadays, we would say that if the side’s length is 1, then the diagonal is $\sqrt{2}$, i.e.: an irrational number). The discovery of such “incommensurable” quantities led to the creation of an innovative concept of proportion about the third century B.C., probably by Eudoxus of Cnidos. Proportions became a main tool of mathematics in general, until well into the XVII century, allowing the comparison of ratios of pairs of magnitudes of the same kind. A Greek proportion, however, is very different from a modern identity, and no concept of equation can be based on it. Thus, for instance, a proportion establishes that the ratio between two segments of line, say A, B, is the same as the ratio between two areas R,S. The Greeks would state this in as strictly verbal fashion. Even shorthand expressions, such as the much later A:B :: R:S, do not appear in Greek texts. The theory of proportions provided significant mathematical results, yet it could not lead to deriving, from the existence of a given proportion, the kind of results derived in modern day equations. Thus, from A:B :: R:S one could deduce that (in modern terms) A+B:A-B :: R+S:R-S. One could not deduce in the same way, however, that A:R::B:S. In fact, it does not even make sense to speak of a ratio between a line A and an area R, to begin with. A main feature of Greek mathematics is that comparisons or simultaneous manipulations can only be made among magnitudes *of the same kind*. This fundamental demand for *homogeneity* would be strictly preserved in all mathematics deriving form Greek sources well until the work of Descartes.

Now, some of the geometrical constructions preformed by Greek mathematicians, and particularly those appearing in Euclid's *Elements*, when suitably translated into modern, algebraic language, appear as instances of establishing algebraic identities, solving quadratic equations, and related issues. Thus for instance, Proposition II.11 of the *Elements*:



To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining square.

In the diagram: Given square ABCD, divide AB at H, so that square AHGF equals rectangle HBDK. If one wants to introduce symbols, AB = a, AH = x, then the problem may be translated into "find x such that $x^2 = (a - x).a$", or the equivalent to a modern second-degree equation.

However, not only symbols of this kind are never to be found in Greek mathematical texts; rather, the whole idea implied in such a translation is alien to the spirit of such texts. In fact, the most essential conceptions on which Greek mathematics is based (and the case of the theory of proportions is only one of them) make it clear that in these works there is no idea of an equation, or even of an unknown quantity that must be calculated by manipulation. Indeed, there is neither symbolic manipulation as such nor well-elaborated, general arithmetical operations that would support the interpretation of such geometrical constructions as a series of "algebraic" exercises cast in the language of geometry, of which the Greeks would presumably have a better domain than of the algebraic one.

In the classical Greek conception of arithmetic, especially as know to us from Books VII-XI of Euclid's *Elements*, a number is a collection of units, namely, only what we would call nowadays a natural number. Negative numbers remain obviously out of this picture, and zero cannot even start to be considered. In fact, even the status of "one" as a number is a matter of ambiguity in certain texts, since it is not really a "collection" as stipulated by Euclid's definition. Such a conception, coupled with the strongly geometrical orientation of Greek mathematics will have long-ranging consequences over the lengthy and involved process that led to the development and full acceptance of more elaborate and flexible idea of number, a fundamental factor in the subsequent development of algebra.

Diophantus

All the above factors taken together imply very clearly that most of the central ideas we commonly associate with algebra are basically absent from Greek classical mathematical thought, though, as we will see, the later eventually and painfully help giving raise to the former. A somewhat different, and idiosyncratic orientation can be found in the work of a later Greek, Diophantus of Alexandria (fl. @ 250 C.E.). Diophantus developed original methods for solving problems that in retrospect may be seen as linear or quadratic (i.e.: second-degree) equations or, even, equations on several variables. In line with the basic conception of Greek mathematics he considered only positive, rational solutions. A problem whose solutions are all negative was called by him "absurd". Diophantus solved specific problems using ad-hoc methods convenient for the problem at hand; he did not provide general methods suitable for some "standard" cases. In general, problems that he solved might actually have more than one (and in some cases even infinite) solutions; yet, he would always stop after finding the first solution. In problems involving quadratic

equations he never suggested that such equations may have two solutions, nor he tried to find the two solutions in particular cases.
Diophantus was the first to introduce some kind of systematic symbolism in addressing problems of this kind. This was more a kind of short-hand writing than real symbols that could actually be freely manipulated. A typical case would be:

$$\Delta^{\nu}\Delta\overline{\beta}\zeta\overline{\delta}\dot{M}\overline{\beta} \pitchfork K^{\nu}\beta\overline{\alpha}^{\nu}\overline{\gamma}$$

(meaning: $2x^4 - x^3 - 3x^2 + 4x + 2$). Here M represents units, ζ the unknown, K^{ν} its square, etc. Since there are no negative coefficients, the terms corresponding to the unknown and its third power, appear to the right of the special symb $\pitchfork$, yet there is nothing like the idea of moving terms form one side of this symbol to the other. Thus, this does not function as the '=' sign of a modern equation. Also, since all Greek letters were used to represent numbers, there was no such possibility as representing "abstract" coefficients in an equation. Thus, Diophantus's use of symbols arises within a framework of rather limited possibilities to begin with.

A typical Diophantus problem would be the following: "To find two numbers such that each, after receiving from the other a given number, will bear to the remainder a given relation".

(In modern terms: $\frac{x+a}{y-a} = r, \quad \frac{y+b}{x-b} = s$)

Diophantus works always with a single unknown quantity ζ. In order to solve this specific problem he takes as given certain values that will allow him a smooth solution: $a = 30$, $r = 2$, $b = 50$, $s = 3$. Now the two numbers sought are $\zeta + 30$ and $2\zeta - 30$, so that the first condition is automatically fulfilled. The second condition is $\zeta + 80 = 3*(2\zeta - 80)$, and by applying his solution techniques one is led to $\zeta = 64$. The two numbers are thus 98 and 94. Some historians see a strong Babylonian influence on this approach, yet no direct evidence exits to support such a claim.

India / China

Indian mathematicians such as Brahmagupta (S VI AD) and Bhaskara (S XII AD) developed non-symbolic, yet very precise, procedures for solving equations of degree one and two, and equations on more than one variable. However, the main contribution of Hindu mathematics to algebra concerns the elaboration of the decimal, positional numeral system, which closely accompanied the development of symbolic algebra in renaissance Europe. By the ninth century the Hindus certainly had a full-fledged decimal, *positional* system, yet many of its central ideas had been transmitted well before that to China and the Islam. Hindu arithmetic, moreover, developed consistent and correct sets of rules for operating with positive and negative numbers, and zero was treated as a number like any other, even in problematic contexts such as division. It would still take several hundreds of years before European mathematics would be in apposition to fully integrate ideas of this kind into the developing discipline of algebra.

Chinese mathematicians during the period parallel to the European middle ages developed their own methods for solving quadratic equations by “radicals” (i.e.: displaying the solutions as expression involving the coefficients, the four basic algebraic operations, and roots of them) and for classifying such solutions. They also attempted to solve higher degree equations in this same direction, yet unsuccessfully. Thus, they were led to approximation methods of high accuracy, such as developed by Yang Hui in the twelfth century AD. The calculational advantages afforded by their expertise with the abacus may help explain why Chinese mathematicians followed more intensively this approach rather than make additional progress with radical methods.

Islamic Contributions

The Islamic tradition in mathematics can be taken to start around A.D. 825, when Muhammad ibn Musa al-Khwarizmi wrote his famous treatise *al-Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala*. By the end of the ninth century a significant mathematical corpus had been already translated into Arabic from the works of Euclid, Archimedes, Apollonius, Diophantus, Ptolemy and others. Likewise, ancient traditions originating in Babylonian and Hindu mathematics, as well as more recent contributions by Jewish sages, were also available to Islamic scholars. This unique background allowed the creation of a whole new kind of mathematics, which implied much more than a mere amalgamation of ideas previously existing in all the traditions mentioned above. A systematic study of methods for solving quadratic equations constitutes a central concern of Islamic mathematicians, and hence their important contributions to the progress of algebraic thinking. A no less central contribution is related to the Islamic reception and transmission of ideas related to the Hindu system of numeration, to which they also added fundamental components lacking so far, such as decimal fractions.

Al-Khwarizmi's work embodies much of what is central to Islamic contributions in this field. He declared his book to be intended as one of practical value, yet this definition hardly applies to what one finds there. In the first part of his book Al-Khwarizmi presented the procedures for solving six types of equations: squares equal roots, squares equal numbers, roots equal numbers, squares and roots equal numbers, squares and numbers equal roots, and roots and numbers equal a square. (ln modern notation: $ax^2 = bx$, $ax^2 = c$,

$bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, and $bx + c = ax^2$, respectively.) Neither zero nor the negative numbers appear here as legitimate coefficients or solutions to equations. Moreover, we find nothing like symbolic representation or abstract symbol manipulation and, in fact in the problems, even the quantities are written in words rather than in symbols. All procedures are described verbally. This is nicely illustrated by the following, typical example:

> What must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: You halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this which is eight, and subtract form it half the number of the roots which is five; the remainder is three. This is the root of the square which you sought for.

In the second part, al-Khwarizmi uses propositions taken from book II of Euclid's *Elements* in order to provide geometrical justification for his procedures. As remarked above, in their original context those were purely geometrical texts. Here they are directly connected, for the first time, to the question of solving quadratic equations. This is a hallmark of the Islamic approach to solving equations: systematize all cases and provide a geometric justification, based on Greek sources. In the XI century we also find it in Ommar Khayyam's application of Greek knowledge on conic sections to questions involving cubic variables.

The use of Greek-style, geometrical arguments in this context also led to a gradual loosening of certain basic, traditional constraints. Thus, Islamic mathematics allowed, and indeed encouraged at variance with the Greek tradition, the unrestricted combination of commensurable and incommensurable magnitudes within the same framework, as well as the simultaneous manipulations of magnitudes of different dimensions as part of the solution of an individual problem. Thus in the work of Abu Kamil the solution of a quadratic equation is a "number", rather than a "line segment" or an "area". Combined with the use of the decimal system, this approach was fundamental in developing a more abstract and general conception of number, which eventually became essential for the creation of a full-fledged abstract idea of an equation.

Early Europe – Leonardo Pisano, Chuquet, Cossists

Greek and Islamic mathematics were basically an "academic" enterprise, having little interaction with day-to-day matters such as building, transportation or commerce. This was to change, as part of important developments in the history of early modern Europe with significant repercussions on the development of algebra as well. The rise of the Italian cities and their expanding trade with the East provide a good example of this, with its growing need for improved methods of bookkeeping, and its fortunate encounter with the Hindu-Arabic numeration, and in particular the use of zero for positional notation in records. In fact, Islamic works on algebra were translated into Latin since the twelfth century, and thus the decimal, positional system was increasingly adopted in places such as Italy, where the important tradition of “abacists” developed.

Leonardo Pisano, an early abacist, wrote his *Liber Abbaci* in 1202. It contained no specific innovation, and it strictly followed the Islamic tradition of formulating and solving problems in purely rhetorical fashion. Yet was instrumental communicating these traditions to the Latin world.

Only in the fourteenth century we find in the work of the abbacists first attempts to introduce abbreviations for unknowns, another important milestone in the way towards full-fledged manipulation of abstract symbols. Thus, for instance, *c* for *cossa* (thing), *ce* for *censo* (square) *cu* for *cubo* (cube), and *R* for *Radice* (root), and even combinations of these for obtaining higher powers. The development of this trend eventually led to works such as Nicolas Chuquet’s *Triparty* (1484), where, as part of a discussion on how to use the Hindu-Arabic numerals, we find relatively complicate symbolic expressions such as

$$R^2\underline{14\,\bar{p}R^2 180} \quad (i.e.\ \sqrt{14+\sqrt{180}})$$

Chuquet also introduced a more flexible way of denoting powers of the unknown, i.e.: 12^2 (for 12 squares) and even $\bar{m}12^{\bar{m}}$ (to indicate $-12x^{-2}$). This is in fact the first time that negative numbers are used in such an explicit way in European mathematics. Chuquet could now write an equation as follows:

$$.3.^2\ \bar{p}.12.\ egaulx\ a\ .9.^1 \quad (i.e.\ 3x^2+12=9x)$$

Like in the Islamic tradition, coefficients are always positive, and thus this is only one of the various cases of possible equations involving squares of the unknown and the unknown itself. Chuquet would say that this is an “impossible equation”, since its solution would involve the square root of –63. We thus find a very illustrative example of the difficulties involved in reaching a more general and flexible conception of number: the same mathematician would allow for the use of negative numbers in a certain context, and even introduce a useful notation for dealing with them, yet at the same time he would completely avoid their use in a different, and still closely connected, context.

In the fifteenth century, the German speaking countries develop their own version of this tradition: the Cossists. It is in the work of Michal Stiffel (1487-1567), Johannes Scheubel (1494-1570), Christoff Rudolff (1499-1545), and others, that we find the first uses of specific symbols for the arithmetic operations, for equality, roots, etc. The subsequent process of adopting such symbols as standard was, nevertheless, a rather lengthy and involved one.

Third and Fourth degree – Cardano et al.

Gerolamo Cardano (1501-1576) was a very famous Italian physician, avid gambler, and prolific writer, with a lifelong interest in mathematics. His widely read *Ars Magna* (1545) contains the most systematic and comprehensive account of XVIth century knowledge on solving third and fourth degree equations. Cardano's presentation follows the Islamic tradition of solving one instance of every possible case and then giving geometrical justifications to all these procedures, based on propositions taken from Euclid's *Elements*. Also, like in the Islamic tradition, coefficients are always positive numbers and the presentation is fully rhetorical with no real symbol *manipulation* as such. And yet, there is an increased *use* of symbols used as shorthand for stating the problems and describing the procedures for solution. Thus, on the one hand we still find here the dominant Greek geometrical perspective: the root of an equation is always a line segment, and the cube, for instance, is the cube built on such a segment. Yet, on the other hand, a cubic equation to be solved will be now written as follows:

cub p: 6 reb aequalis 20

[i.e.: $x^3+6x = 20$]
and the solution is presented as :

R. V: cu.R. 108 p: 10 m: R. V: cu. R. 108m: 10.

(meaning $x = \sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$)

Not viewing negative numbers as possible coefficients of the equations prevented the development of the notion of a general third-degree equation, and thus Cardano dealt with all thirteen possible cases of third-degree equations and also with 20 different cases of fourth-degree equations, following for the latter the procedures developed by his student Ludovico Ferrari (1522-1569). On the other hand, he was willing to consider, in some places, the possibility of negative (or "false") solutions. This allowed him formulate some general rules, such as the claim that in equations with three real roots (including even negative ones) their sum equals the coefficient of the squares.

In spite of this basic acceptance of traditional views on numbers, the very outcome of Cardano's work compelled him to add flexibility to them. Thus, for instance, in the following problem: to divide 10 into two parts whose product is 40. The answer, obtained by well-known methods, is $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Cardano's attitude towards this solution was one of uneasiness, but he finally accepted it, declaring it to be "as refined as it is useless". It was only Rafael Bombelli (1526-1572) who undertook a more systematic treatment of them in 1572.

Besides the specific contribution of Cardano and some of his contemporaries to systematically solving equations of third and fourth degree we find here significant, additional advances toward a broader acceptance and full legitimation of new kinds of numbers beyond the rational positives.

Viète

It is in the work Françoise Viète (1540-1603) that we find the first consistent, coherent and systematic conception of an algebraic equation in the modern sense of the word. Viète was a prominent lawyer with a keen interest in mathematics and highly developed cryptographic skills that he put to use in the service of King Henri III.

A main innovation of Viète's book *In artem analyticam isagoge* (1591) is the use of well-chosen symbols of one kind for the unknowns (vowels), and of another kind for the known quantities (consonants). This allowed not only flexibility and generality in solving linear and quadratic equations, but also the introduction of a real key point absent from all its predecessor's work, namely, a clear analysis of the relationships between the forms of the solutions and the values of the coefficients of the original equation. Viète saw his contribution as that of developing a "systematic way of thinking" leading to general solutions rather than just a "bag of tricks" variously helping to solve specific problems.

By combining existing usage with his own innovations, Viète was able to clearly formulate equations and to provide rules for transposing factors from one side to the other in order to find the solutions. An example of such an equation would be:

A *cubus* + C *plano in* A *aequatus* D *solido*
(modern terms: $x^3 + cx = d$)

And a rule would state that:

$$\frac{Z\ plano}{G} + \frac{A\ plano}{B}\ aeq.\ \frac{G\ in\ A\ plano + B\ in\ Z\ plano}{B\ in\ G}$$

Notice that each of the terms involved here is of "dimension" 1. On the left-hand side, we have, e.g. the 2-dimensional magnitude *Z plane* "divided by" the 1-dimensional one *G*. On the right-hand side, we have a sum of two 3-dimensional magnitudes divided by a product of two 1-dimensional ones. Thus, Viète did not break the important Greek tradition whereby the terms equated must always be of the same "dimension". Yet for the first time it became possible, in the framework of an equation, to multiply or divide both sides by a certain magnitude. The result is a new equation homogeneous in itself, yet not homogeneous with the original one.

Viète showed how to transform given equations into others, already known (e.g., in modern notation,: $x^3+ax^2=b^2x \implies x^2+ax=b^2$). Thus, he reduced the number of cases of cubic equations, from the 13 given by Cardano and Bombelli. Yet, since he used no negative or zero coefficients, he did not yet generalize *all* the possible cases into a single one.

Viète applied his methods to solve, in a general, abstract-symbolic fashion, problems similar to those in the Diophantic tradition. However, very often he also rephrased his answers in plain words, as if to reassure his contemporaries, and perhaps even himself, of the validity of these new methods.

Numbers

If the work of Viète may be said to contain a clear, systematic, and coherent conception of the notion of equation, that served as a broadly accepted starting point for later developments in this field, no similar single point of reference can be mentioned regarding a general conception of number.

Some significant milestones may nevertheless be mentioned in this context, and prominent among them is an influential booklet published in 1585 by the Flemish engineer Simon Stevin (1548-1620). *Le Disme* was intended as a practical manual aimed at teaching the essentials of operating with decimal fractions, but it also contained many conceptual innovations. This is the first mathematical text where the all-important separation between "number" and "magnitude", going back all the way to the Greeks, was explicitly and totally abolished. Likewise, Stevin declared that 1 is a number just like any other, and that the root of a number is a number as well. Stevin also showed how one and the same single idea of number, expressed as decimal fractions, could be equally used in such separate context as land surveying, volume measurement, astronomical and financial computations. The very need for an explanation of this kind helps us realize how far were Stevin's contemporaries and predecessors from the abstract notion of numbers we are used to nowadays, and that his booklet did much for consolidating and spreading.

And yet, by the end of the sixteenth century and throughout the seventeenth century there were still lively debates among mathematicians about the legitimacy of using the various kinds of numbers. For example, concerning the irrationals, some prominent mathematicians (Pascal, Barrow, Newton) were willing to grant them legitimacy only as geometrical magnitudes. The negative numbers were sometimes seen as even somewhat more problematic, and in many cases negative solutions of equations continued to be considered as "absurd" or as "devoid of interest" among many. Finally, the complex numbers, even though Bombelli had given precise rules for their arithmetic, were still ignored by many mathematicians. Descartes, for one, rejected them totally.

Towards the eighteenth century all these discussions dwindled away and a new phase in the development of the concept of number began, whereby the systematization and the search for adequate foundations for the various systems was initiated.

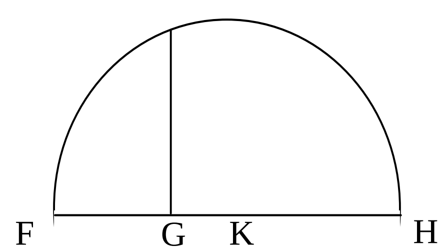
Part II. The classical discipline of algebra

The classical discipline of algebra starts its actual development after the consolidation of the idea of an equation in Viète's work. This development, to be described in the present section, comprises several related trends, among which the following deserve special attention: (1) the quest for systematic solutions of higher order equations, including approximation techniques; (2) the rise of polynomials and their study as new, autonomous mathematical entities, and the development of a theory of polynomial forms; (3) the increased adoption of the algebraic perspective in other mathematical disciplines (geometry, analysis, logic, etc.). At the same time, during this same period of time, new mathematical objects gradually arose (groups, rings, fields, etc.), that eventually came to replace the study of polynomials as the main subject matter of algebra and became the new focus of interest of the discipline (as will be described in Part III).

Fermat and Descartes: Analytic geometry, theory of polynomials

The creation of what came to be known as analytic geometry is usually attributed to two famous French thinkers: Pierre de Fermat (1601-1665) and René Descartes (1596-1650). They used the algebraic techniques developed by Cardano and Viète and applied them to tackle classical geometrical problems that had remained unsolved since the time of the Greeks. The new kind of organic connection between algebra and geometry thus established meant a major breakthrough without which the subsequent development of mathematics in general, and in particular of geometry and the calculus, would be unthinkable. It also had significant impact on algebraic thinking.

In his famous book *La Géométrie* (1637) Descartes established equivalences between algebraic operations and geometrical constructions. In order to do so, he introduced a "unit length", serving as reference for all other lengths and all operations among them. Thus, for instance, given a segment GH, one is asked to find its square root. Descartes draws the straight line FH, where FG is taken to be "equal to unity"



Bisect FH at K, draw the circle FIH about K, and draw the perpendicular IG on G. Elementary properties of the circle, as described e.g., in Euclid's *Elements*, imply that the square built on IG equals the quadrilateral built on FG, GH. Thus, in what might appear to be an ordinary application of classical Greek techniques, Descartes has found the square root of any given number, as represented by a line segment. However, the key step in the construction has been the introduction of the "unit length" FG. This seemingly trivial move, or anything similar to it, had never been part of Greek geometry and its legacy and, of course, it had enormous repercussions on what could now be done by applying algebraic reasoning to geometry. As part of his contributions in this context, Descartes also introduced a notation that allowed great flexibility in symbolic manipulation. For instance, he would write

$$\sqrt{C.a^3 - b^3 + abb}$$

to denote for the cubic root of this algebraic expression. This was a direct continuation (with some improvement) of techniques and notations introduced by Viète. Yet Descartes also introduced a new idea with truly far-reaching consequences when he explicitly eliminated the demand for homogeneity among the terms appearing in any equation (although, for convenience he tried to stick to homogeneity wherever possible).

Descartes' program was based on the idea that certain, well-known geometrical loci (straight lines, circles and conic sections) can be characterized in terms of specific kinds of equations involving magnitudes that are taken to represent line segments. This program, however, did not

envision the no less important, reciprocal idea, namely, that of finding the curve corresponding to an arbitrarily given algebraic expression. Descartes was also aware that much information about the properties of a curve (area, tangents, etc.) could be derived from its equation, yet he did not elaborate this direction in any detail.

On the other hand he was the first to discuss separately and systematically the algebraic properties of polynomials equations. This includes the correspondence between the degree of an equation and the number of its roots, the factorization of a polynomial with known roots into linear factors, the rule for counting the number of positive and negative roots of an equation, and the method for obtaining a new equation having its roots equal to those of a given equation, but increased or diminished by a given quantity.

Gauss: the fundamental theorem of algebra

Descartes' work was a starting point for the definite transformation of polynomials into an autonomous object of intrinsic mathematical interest. Algebra became identified, to a large extent, with the theory of polynomials. A clear notion of a polynomial equation, together with existing techniques for solving some of them, allowed for a coherent and systematic reformulation of many questions that mathematicians in the past had dealt with in a more haphazard fashion. High in the agenda remained the open question of finding algebraic solutions of equations of degree higher than four. Closely related to this was the question of the kinds of numbers that should count as legitimate roots of equations. The attempts to deal with these two important problems (see below) helped realize the centrality of another pressing question that needed to be elucidated, namely, the questions of the number of solutions that a given polynomial equation has.

The answer to this question is afforded by the so-called Fundamental Theorem of Algebra (FTA), which asserts that every polynomial in real coefficients can be expressed as the product of linear and quadratic (real) factors, or, alternatively, that every polynomial equation of degree n in complex coefficients has n complex roots.

Leading mathematicians such as Leibniz, Euler, D'Alembert, and Laplace, attempted throughout the eighteenth century to provide proofs of statements variously equivalent to FTA. The flaws contained in their proofs were evidently related to remaining unclarities still affecting the main concepts involved: polynomials and the status of the various number systems. Indeed, the process of criticism and revision that accompanied the successive attempts to formulate and prove some correct version of the FTA effectively contributed to a deeper understanding of both the specific properties of polynomials as part of the then emerging theory of continuous functions, and the general behavior of the system of complex numbers. It is in this context, more than in any other one perhaps, that the need for a full legitimation of the complex numbers became necessary, in order to allow for a coherent basis that justified the existence of n roots in the general case.

The first complete proof of the FTA is usually attributed to Carl Friedrich Gauss (1777-1855) in his doctoral dissertation of 1799. Subsequently, Gauss himself provided three additional proofs. Later on, additional proofs were given by others, such as the Swiss bookkeeper Jean-Robert Argand (1768-1822) in 1814, and the German mathematician father and son, Hellmuth Kneser (1898-1973) in 1940 and Martin Kneser in 1981.

A remarkable fact of all these proofs is that they are based on methods and ideas that are usually considered as "analytical" or "topological", and thus foreign to "algebra" proper. Mathematically, the theorem is "fundamental" in that it establishes an essential property of the most basic concept around which the discipline as a whole is built. In this very sense, however, the theorem was also fundamental from the historical point of view, since it contributed to the consolidation of the discipline, its main tools and its main concepts.

Worth of mention is also the fact that in the attempts to prove the theorem, both Argand and Gauss were led to elaborate in detail their interpretations of the complex numbers as oriented segments on the plane (see below).

Impasse with radical methods: Lagrange, Ruffini, Gauss, Abel

A major breakthrough in the way to elucidating the question of algebraically solving higher-degree equations was achieved by Lagrange in 1770. Rather than trying to directly find a possible solution for an equation of degree five, Lagrange attempted to clarify first *why* all attempts to do so had failed so far. He investigated the known solutions of cubic (i.e.: third-degree) and cuartic (i.e.: fourth-degree) equations and in particular, how certain algebraic expressions connected with those solutions remain invariant when the coefficients of the equations are permuted with one another. Lagrange was certain that a deeper analysis of this invariance would provide the key insight to understanding the essence of existing methods of solution by radicals, in the hope of being then able to extend them successfully to higher degrees.

Using the ideas developed by Lagrange, the Italian Paolo Ruffini (1765-1822) was the first mathematician ever to assert the impossibility of an algebraic solution for the *general* polynomial equation of degree greater than four. He adumbrated in his work the notion of a group of permutations (see below), and worked out some of its basic properties. Ruffini's proofs, however, contained several, significant gaps.

Parallel to this, and in a somewhat contrary direction, between 1796 and 1801, in the framework of his seminal number-theoretical investigations, Gauss systematically dealt with the so-called cyclotomic equations, $x^p-1 = 0$ ($p>2$, prime), and developed new methods for solving these *particular* cases of higher-order polynomial equations.

The Norwegian mathematical star of the early nineteenth century, Niels Henrik Abel (1802-1829), provided in 1824 the first clear and accepted proof of the impossibility of solving by radicals equations of degree five or above. This did not bring the question to an end, but rather opened an entirely new field of research, since, as Gauss's example showed, *some* equations were indeed solvable. In 1828, Abel suggested two main points for research in this regard: (1) to find all equations of a given degree solvable by radicals; (2) to decide if a given equation can be solved by radicals. His early death in complete poverty, two days before receiving an announcement at home that he had been appointed professor in Berlin, prevented Abel of undertaking this program, as well as many other research plans he had conceived, especially in analysis.

Galois's ideas and their gradual spread

Rather than establishing for specific equations if they can or cannot be solved by radicals, as Abel had suggested, Evariste Galois (1811-1832) pursued the somewhat more general problem of defining necessary and sufficient conditions for the solvability of any given equation. Galois' short, and exceptionally turbulent, life has been the subject of many books and films, and countless myths have been weaved around it. There can be no doubt, at any rate, that his work was a truly major force leading to a total reshaping of the discipline of algebra, as well as to other significant advances in additional fields of mathematics.

Prominent among these seminal ideas was the clear realization how to formulate precise solvability conditions in terms of the properties of a group of permutations associated with the polynomial in question. The following example may help clarify the ideas involved here. A permutation of a set, say the set of three letters *a*, *b*, and *c*, is any re-ordering of the elements, and it is usually denoted as follows:

$$P = \begin{pmatrix} a & c \\ b & a \\ c & b \end{pmatrix}$$

This permutation takes *a* to c, *b* to *a*, and *c* to *b*. In this case, there are six different ways to establish such permutations. Now, it turns out that one can operate with such permutation, and combine any two of them into a third one. Thus for instance:

$$\begin{pmatrix} a & c \\ b & a \\ c & b \end{pmatrix} * \begin{pmatrix} a & a \\ b & c \\ c & b \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \\ c & c \end{pmatrix}$$

Here *a* goes first to *c*, and then from *c* to *b*, etc. This operation is associative: namely given three permutations *P*, *Q*, *R* , one has: (*P***Q*)**R* = *P**(*Q***R*). Also, there is one permutation, which is neutral with respect to the operation, namely the identity permutation

$$I = \begin{pmatrix} a & a \\ b & b \\ c & c \end{pmatrix}$$

Finally, for each permutation there exists a second one that, when combined with the first, yields the identity permutation.

The concept of abstract group, that was developed somewhat later (see below), consistes of a set of abstract elements on which an operations is defined and which satisfies the three above mentioned conditions: associativity, existence of a single neutral element, and existence of an inverse element for each element in the set.

This abstract notion is not fully present in Galois's work. Like some of his predecessors mentioned above, Galois focused on the group of permutations of the roots of the equation. His specific contribution was to define a well-elaborated "reduction" process over this group, and to base the criteria of solvability on the outcome of such a process. The idea was to start with certain "rationally known" quantities for the equation (roughly:

quantities that can be produced by combining the coefficients of the equation by means of the four standard arithmetic operations, together with root extraction). This initial set is then enlarged by gradually adjoining combinations of roots obtained from auxiliary equations introduced to reduce the degree of the original equation (the kind of equations that Cardano, for instance, had introduced for solving cubic and quartics). In the enlarged set there are fewer expressions that remain invariant under permutation of the roots. Thus, the associated group is gradually "reduced", and, as already said, the possible outcomes of such a reduction process would determine the solvability conditions for the original equation. Galois' results, it must be stressed, refer to existence conditions; they do not provide ways to calculate the radical expressions of the solutions in those cases where they exist.

Galois' work can be seen both as the culmination of a main line of development in the history of algebra -- solving equations -- and as the beginning of a second line -- the study of abstract structures. Work on permutations, started by Lagrange and Ruffini, had received further impetus with the contributions of the leading French mathematician Augustin Louis Cauchy (1789-1857), beginning in 1815. In a later work of 1844, he systematized much of this knowledge and introduced basic concepts such as order, conjugation and the so-called cycle notation. For instance the permutation

$$\begin{pmatrix} a & b \\ b & a \\ c & e \\ d & c \\ e & d \end{pmatrix}$$

was denoted by Cauchy as (*ab*)(*cde*), meaning that the permutation is obtained by one substitution (*a* by *b*) and one cycle (involving *c*,*d*,*e*).

Galois gave prominent place to concepts such as normal subgroups and irreducible equations. The impossibility of solving the general quintic by radicals followed from his general arguments and from the fact that the specific group associated to it has no proper, normal sub-groups.

A series of unusual and unfortunate events involving the most important French contemporary mathematicians prevented Galois' ideas form being published for a long time. It was not until 1846 that Joseph Liouville (1809-1882) edited and published for the first time, in his prestigious *Journal de Mathématiques Pures et Appliquées*, the important memoire where Galois had presented his main ideas and that the Paris Academy had turned down in 1831. Liouville also lectured in Paris on the topic, to a reduced audience. Leopold Kronecker (1823-1891) working in Berlin, applied some of these ideas to number theory in 1853, and Richard Dedekind (1831-1916) lectured on Galois theory in 1856 at Götitngen. At this point, however, the impact of the theory was still minimal.

A major turning-point came with the works of the leading Paris mathematician Camille Jordan (1838-1922) who published a series of papers and an influential book in 1870. Jordan elaborated a theory of groups of permutations, independently of any reference to equations, and the use of this theory to the question of algebraic solvability appeared in his book just as a particular application of the theory. A lengthy process eventually led from here to the conception of Galois theory as the study of the interconnections between extensions of fields and the related Galois groups of equations, a conception that would prove fundamental for developing a completely new approach to algebra in the 1920s. Major contributions to the development of this point of view for Galois theory came variously from later works by Dedekind, Enrico Betti (1823-1892), Henrich Weber (1842-1913), and Emil Artin (1898-1962), among others.

Groups in geometry and number theory (350)

Galois theory arose in direct connection with the study of polynomials, and thus the notion of group developed from within the main-core of classical algebra. However, it did also find important, early applications in other mathematical disciplines throughout the nineteenth century, particularly geometry and number theory. This pervasiveness no doubt contributed to the increased interest it awoke among mathematicians at large, and to the general shift of disciplinary focus it eventually helped bring about in algebra.

Felix Klein (1849-1925) was still a very young professor when in his inaugural lecture at the University of Erlangen (1872) he suggested how group theoretical ideas might be fruitfully put to use in the context of geometry. Since the beginning of the 19^{th} century the study of projective geometry had attained renewed impetus, and later on, non-Euclidean geometries were introduced and increasingly investigated. This proliferation of geometries raised pressing questions concerning both the interrelations among them and their relations with the empirical world.

Klein suggested that the many kinds of existing geometries could be classified and ordered within a conceptual hierarchy: thus, for instance, projective geometry seems to be more fundamental, because projective properties are relevant also, e.g., in Euclidean geometry. The main concepts of the latter, however, such as length or angle, have no significance in the former. But then, this hierarchy may be expressed in terms of transformations that leave invariant such properties as are distinctly relevant to each of the geometries in question. These transformations, it turns out, are best understood when seen as forming a group. An example related with Euclidean geometry clearly illustrates the basic idea behind this.

By rotating any figure on the plane, none of its Euclidean properties is affected. One can easily define an operation on the set of all rotations of the plane: if rotation I rotates the plane by an angle α, and rotation J by an angle β, then rotation $I*J$ rotates it by an angle $\alpha+\beta$. This operation is obviously associative. The neutral element is the rotation associated with angle 0°, and the inverse of the rotation associated with angle α is that associated with angle –α. Thus the group of rotations of the plane is a group of invariant transformations for Euclidean geometry. The groups associated with other kinds of geometries may be somewhat more involved, but the idea remains the same. Klein's idea was that the hierarchy of geometries might be reflected into a hierarchy of groups, whose properties might turn to be easier to study and to grasp.

In the 1880's and 1890's, Klein's friend, the Norwegian Sophus Lie (1842-1899) undertook, together with some of his students at Leipzig, the enormous task of classifying all possible groups of continuous groups of geometric transformations, a task that would eventually evolve into the modern theory of Lie groups and Lie algebras. At roughly the same time, Jules Henri Poincaré (1854-1912) studied in France the groups of motions of rigid bodies, a work that contributed more than the others mentioned here to spreading the notion of group as a main tool in modern geometry.

The notion of group also started to appear prominently in number theory in the nineteenth century, especially in the work of Gauss on modular arithmetic. In this context he proved results that were later generally reformulated in the abstract theory of groups. Thus, for instance (in modern terms), that in a cyclic group there always exists a subgroup of every order dividing the order of the group. Gauss also studied the group-theoretical properties of transformations of quadratic forms, forms that play a major role in his number-theoretical investigations.

Arthur Cayley (1821-1895), one of the most prominent British mathematicians of his time, was the first to explicitly realize, in 1854, that a group could be defined abstractly, i.e.: without any reference to the nature of its elements and only by specifying the properties of the operation defined on them. Generalizing on Galois' ideas, Cayley took a set of meaningless symbols 1,α,β, … with an operation defined on them.

		α	β	…
		α	β	…
α	α	α^2	$\alpha\beta$	…
β	β	$\beta\alpha$	β^2	…
…	…	…	…	…

Cayley demanded only that the operation be closed with respect to the elements on which it is defined, but he assumed implicitly that it is associative and that each element has an inverse. He thus deduced correctly some basic properties of the group, such as for instance, that if the group has *n* elements, then, for each element θ one has $\theta^n=1$.

In 1854, even the idea of group of permutations was rather new and thus Cayley's work had little impact. It would take until 1882, and several additional articles by Cayley himself, as well as by Eugene Netto (1846-1919) and Georg Frobenius (1849-1917), before Walther van Dyck (1856-1934) would publish in 1882 the full-fledged and most general definition of an abstract group. Books like Heinrich Weber's *Lehrbuch de Algebra* (1895) and *Theory of Groups* (1897) by William Burnside (1852-1927) were instrumental in bringing the theory to a truly broad audience of mathematicians.

Dedekind: Ideals and Fields

Some additional, new concepts that eventually became prominent in algebra have their origin in nineteenth-century work on number theory. Important examples of this are fields and ideals that arose in the work of Dedekind in connection with the attempts to generalize the so-called theorem of unique prime factorization (TPF). The TPF asserts that every natural number can be written as a product of its prime factors in a unique way, except perhaps for the order (e.g: 24 =2.2.2.3). This property of the natural numbers was well-known to mathematicians, at least implicitly, since the times of Euclid.

The attempts to generalize this theorem arose as soon as nineteenth century mathematicians started to explore new numerical domains, especially those associated with the widespread acceptance of the complex numbers into mainstream mathematical discourse. Once should not be surprised, then, to find the name of Gauss in this context. In his classical investigations on arithmetic, and in particular the so-called problem of "higher reciprocity", Gauss was led to investigate the factorization properties of numbers of the type a+*i*b (a,b integers; $i^2 = -1$), sometimes called "Gaussian integers". In doing so, Gauss was not only using *complex* numbers in order to solve a problem directly relevant to higher arithmetic of *ordinary* integer numbers, a fact remarkable in itself, but he was also opening the way to the detailed investigation of special sub-domains of the complex numbers.

In 1832 Gauss proved that the Gaussian integers satisfy a generalized version of the TPF, where the "prime factors" had to be especially defined in this domain. In the 1840s Ernst Edward Kummer (1810-1893) started work on further generalizing these results to other, even more general domains of complex numbers, such as those of numbers a+θb, where $\theta^2=n$ (*n* a fixed, positive or negative, integer number), or numbers a+ρb, where $\rho^n=1$, $\rho \neq 1$, $n>2$ (these are called domains of cyclotomic integers). Although Kummer did prove interesting results, by introducing what he called "ideal factors", it finally turned out that the TPF is not generally valid in such general domains. The difficulty is easily seen in the following example from the domain of numbers a+b $\sqrt{-5}$. Consider the factorization of 21:

$$21= 3.7 = (4 + \sqrt{-5}) * (4 - \sqrt{-5})$$

It can be showed that neither of the numbers 3,7, 4 ± √-5 can be written as a product of two different numbers in this same domain. Thus, in one sense they are "prime". However, at the same time they violate a property of prime numbers known from the time of Euclid, namely:

if a prime number p divides a product ab, then it either divides a or b (P)

Here, for instance, 3 divides 21 which is the product of =4 + √-5 and 4 - √-5, but does not divide either of the factors.

This situation led to the introduction of a new concept: indecomposable numbers. In classical arithmetic any prime is also indecomposable, but in more general domains, such as here, 3 is indecomposable, yet not prime in the sense of (P).

The question thus remained open of what are the actual domains of validity of the TPF and how should its generalized version be properly formulated. This problem was undertaken by Richard Dedekind in a series of works spanning over 30 years and starting in 1871. Dedekind's general methodological approach promoted the introduction of new concepts around which entire theories could be built. Specific problems would then have to be solved as instances of the results afforded by the general theory. Among the general concepts he introduced for dealing with the problem of generalizing the TPF were fields and ideals

A main question pursued by Dedekind was the precise identification of those subsets of the system of complex numbers on which it makes sense to attempt generalized formulations of the TPF. The first step towards answering this question was to define fields, namely, any subset of the complex numbers that is closed under the four basic arithmetical operations (except division by zero). The largest of these fields is the whole system of the complex numbers, whereas the smallest one is that of the rational numbers. As was the case with groups, the concept of field will eventually evolve into a fully abstract one, in which not the nature of terms is relevant, bur rather the properties of the operations defined on them.

Now, in ordinary arithmetic the integers are a distinguished set inside the real numbers, that satisfies, among others, the TPF. Using the concept of field and some other, derivative ones, Dedekind could identify, within the relevant sub-fields of the complex numbers, what is the precise collection of numbers into which the same theorem would be extended. These are the algebraic integers of the field in question.

Finally, Dedekind introduced the ideals, which elaborated on the idea of "ideal number" formerly defined by Kummer. A main methodological trait of Dedekind's innovative approach to algebra was to translate ordinary arithmetic properties into properties of sets of numbers. In this case, he focused on the set *I* of multiples of any given integer, and pointed out two of its main properties:

1. if *n*,*m* are two numbers in *I*, then their difference is also in *I*

1. if *n* is a number in *I* and *a* is any integer, then their product is also in *I*

As he did in many other contexts, Dedekind took these *properties* and turned them into *definitions.* An ideal in the domain of algebraic integers of any given field of complex numbers, is any collection of such integers that satisfies properties (1) , (2) above. This is the concept that allowed him to generalize the TPF in distinctly set-theoretical terms.

In fact, in ordinary arithmetic, the ideal generated by the product of two numbers equals the intersection of the ideals generated by each of them. Thus for instance, the set of multiples of 6 (i.e.: the ideal generated by 6) is the intersection of the ideal generated by 2 and that generated by 3. His generalized versions of the TPF were phrased precisely in these terms for general fields of complex numbers and their ideals. Dedekind distinguished among different types of ideals and different types of decompositions, but the generalizations were all-including and precise. And more importantly: What was originally a result on numbers, their factors and their products, ended up being reformulated as a result on special domains, special sub-sets of numbers, and their intersections.

Dedekind's results were not only important for a deeper understanding of the important question of unique prime factorization. They were also instrumental in helping bring about a deep shift of focus that implied, in the long run, a major change in the most basic conceptions about the scope and subject-matter of algebra. Dedekind not only introduced, here and in some other places, the set-theoretical approach into algebraic research, but he also defined at the same time some of the important concepts that would later become the hard-core of modern algebra: fields, modules, rings, lattices, etc. Moreover, Dedekind's ideal-theoretical approach was soon successfully applied to the question of factorization of polynomials as well, thus connecting itself once again to the main focus of research of the classical discipline of algebra.

Determinants, Matrices, British Symbolic Algebra

In spite of the many novel ideas that arose in connection with algebra in the nineteenth century, solving equations and studying properties of polynomial forms continued to be the main focus of interest of the discipline. An important offshoot of the study of polynomials was the development of the theory of algebraic invariants, to which much effort was dedicated by leading algebraists since the 1840s, especially in Germany (but which, for lack of space will not be considered here). The study of systems of equations led to developing the notion of a determinant and, later on, to the theory of matrices.

Given a system of *n* linear equations in *n* unknowns, a determinant is the result of a certain combined multiplication and addition of the coefficients involved, that allows calculating directly the values of the unknowns. Thus for instance, given the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The determinant of the system is the number $\Delta = a_1.b_2 - a_2b_1$, and the values of the unknowns are given by

$$x = (c_1.b_2 - c_2.b_1)/\Delta \qquad y = (a_1.c_2 - a_2.c_1)/\Delta$$

Historians coincide in pointing out the contribution of the Japanese mathematician Seki Kowa (1642-1708) as the earliest, systematic use of methods of this kind. In Europe, credit is usually given to Gottfried Wilhelm von Leibniz (1646-1716) at roughly the same time, while Etienne Bézout (1730-1783) and Alexandre-Theophile Vandermonde (1735-1796) appear as contributing to the continued development of related ideas during the eighteenth century.

Cauchy published in 1815 the first truly systematic and comprehensive study of determinants (including the very name). He introduced the notation $(a_{1,n})$ for the system of coefficients of the system and showed how to calculate the value of the determinant by expanding any row or column with the adjoint of every element.

Closely related with determinants is the idea of a matrix, namely, any arrangement of numbers in lines and columns. That such an arrangement can be taken as an *autonomous mathematical object*, on which one can define a special arithmetic and operate as with ordinary numbers, was first conceived by Cayley and his good friend James Joseph Sylvester (1814-1897), in the 1850s. Determinants were a main, direct source for this idea, but so were ideas contained in previous work on number theory by Gauss and by Ferdinand Gotthold Eisenstein (1823-1852).

Given a system of linear equations:

$$\begin{aligned}
\xi &= \alpha x + \beta y + \gamma z + \dots \\
\eta &= \alpha' x + \beta' y + \gamma' z + \dots \\
\zeta &= \alpha'' x + \beta'' y + \gamma'' z + \dots \\
\dots &= \dots + \dots + \dots + \dots
\end{aligned}$$

Cayley represented it with a matrix as follows:

$$(\xi, \eta, \zeta, \dots) = \begin{pmatrix} \alpha & \beta & \gamma & \dots \\ \alpha' & \beta' & \gamma' & \dots \\ \alpha'' & \beta'' & \gamma'' & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} (x, y, z, \dots)$$

The solution can then be written as:

$$(x,\ y,\ z,\ \ldots) = \begin{pmatrix} \alpha & \beta & \gamma & \ldots \\ \alpha' & \beta' & \gamma' & \ldots \\ \alpha'' & \beta'' & \gamma'' & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}^{-1} (\xi,\ \eta,\ \zeta,\ \ldots)$$

The matrix bearing the –1 exponent is called the "inverse" matrix, and, of course, the whole difficulty of this approach lays in calculating it. Cayley showed how to do so using its determinant. Once this matrix is calculated, then the arithmetic of matrices has allowed to solve a system of equations by a simple analogy with the case of a linear equations: $b = a.x \Rightarrow x = a^{-1}.b$.

Cayley was soon joined by other mathematicians, such as William Rowan Hamilton, Frobenius, and Jordan in developing the theory of matrices, which gradually became a very useful tool in many mathematical domains, ranging from analysis, to geometry and to the emerging discipline of linear algebra. But the importance of this notion for the development of algebra is also connected with the fact that matrices continued to enlarge the range of new notions that, taken together, would come to change the whole conception of the discipline (see Part III). Moreover, matrices embodied a new, mathematically significant instance (though not the first known one) of a system with a well-elaborate arithmetic, whose rules nevertheless *departed* from the traditional ones, in the important sense that multiplication would not necessarily be commutative.

In fact, early developments of matrix theory are naturally connected with a central trend developed in English mathematics after 1830 by mathematicians such as George Peacock (1791-1858) and Augustus de Morgan (1806-1871). In trying to overcome the last remnants of the debates around the legitimacy of the uses of negative and imaginary numbers, these mathematicians suggested that algebra be conceived as a purely formal, symbolic language, irrespective of the nature of the objects whose laws of combination it stipulates. This view allows, in principle, for new kinds of arithmetic, differing from ordinary one in some respects, such as is the case with matrices. The British tradition of symbolic algebra was instrumental in bringing about an eventual shift in the focus of algebra, from trying to understand the nature of its *objects* (numbers, polynomials, or whatever), to elaborating the properties of *operations* of any kind. Still, in most respects Peacock and De Morgan strove after gaining a deeper understanding of the objects of classical arithmetic and algebra, rather than launching a new discipline, where the emerging notion, and the various instances, of abstract algebraic structures would become the center of interest.

Another important offshoot of this school, which must be mentioned here, is the development by De Morgan himself, and by George Boole (1815-1864) of an elaborate "algebra of logic". This innovative work, together with its somewhat later parallel in Germany in the work of Ernst Schröder (1841-1902), were instrumental in transforming the discipline of logic from a purely metaphysical into a mathematical one. It also added to the growing understanding of how far can algebraic thinking be potentially taken, away form its narrow conception as a discipline dealing only with polynomial equations and forms, as well as with operations only with numbers of various kinds.

Quaternions, Vectors, Hypercomplex Systems

Remaining doubts about the legitimacy of complex numbers were finally dispelled when the geometrical interpretation became widespread among mathematicians. This interpretation, initially conceived by the Norwegian surveyor Caspar Wessel (1745-1818), and by Argand, was made known to a larger audience of mathematicians mainly through its explicit use by Gauss in his 1848 proof of the FTA. Under this interpretation, every complex number appears as a directed segment on the plane, which is characterized by its length and its angle of inclination with respect to the *x*-axis. The number *i* thus corresponds to the segment of length 1 which is perpendicular to the *x*-axis. Once a proper arithmetic is defined on these numbers it turns out that $i^2 = -1$, as expected.

A second, alternative interpretation that eventually became widely accepted was published in 1837 by the Irish versatile mathematician and physicist Sir William Rowan Hamilton (1805-1865), very much within the spirit of the British school of symbolic algebra. Hamilton defined a complex number as a pair (α,β) of real numbers, and he also stated the laws of arithmetic for such pairs. Multiplication, for instance, would be defined as follows:

$$(\alpha,\beta)\,(\chi,\delta) = (\alpha\chi - \beta\delta,\ \beta\chi + \alpha\delta)$$

Now the root of –1 is defined as (0,1), and indeed the above rule implies that multiplying this number by itself we obtain (-1,0). This formal interpretation obviates the need to give any “essentialist” definition of what the complex number are.

Starting in 1830 Hamilton pursued intensely, and unsuccessfully, the task of extending this scheme into triplets (*a,b,c*), in an attempt to develop a new mathematical tool that he expected to be of great utility in physics. The difficulty lay in consistently defining a multiplication for such a system, which in hindsight is known to be impossible. Yet, in 1843 Hamilton realized that the generalization he was looking for had to be found in the system of quadruplets (*a,b,c,d*). He wrote them, in analogy with the complex numbers, as: $a + bi + cj + dk$, and his new arithmetic was based on the rules: $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$ and $ik = -j$. This was indeed the first example of a coherent, significant mathematical system that preserved all the laws of ordinary arithmetic, except commutativity.

In spite of Hamilton’s initial hopes, quaternions did not actually become really useful in physics. Nevertheless, his ideas did enormously influence the gradual introduction and use of *vectors* into physics. Hamilton himself named the “real” part *a* of the quaternion scalar, and the “imaginary” part $bi + cj + dk$ vector, and defined what is nowadays known as scalar and vectorial product. It was through the successive work of the British Peter Guthrie

Tait (1831-1901), James Clerk Maxwell (1831-1879), and Oliver Heaviside (1839-1903), and of Josiah Willard Gibbs (1839-1903) at Yale, that an autonomous theory of vectors was first established while developing on Hamilton's initial ideas.

Quaternions, nevertheless, remained important *inside* mathematics, mainly as part of the increased interest in all the new ideas that have been emerging throughout the century in algebra, and that have been described above. Indeed, this was by no means the only attempt at generalizing the now stabilized idea of complex numbers: starting in 1844 with the work of Hermann Grassman (1809-1877), it is possible to mention a long list of mathematicians whose works can be gathered under the common title of "hypercomplex systems" of all kinds, that would eventually find applications in the new algebraic developments of the beginnings of the twentieth century.

Weber's *Lerhbuch der Algebra*

Heinrich Weber's *Lehrbuch der Algebra* was published in three volumes and saw several editions starting in 1895. This was the last classical textbook in the discipline in the nineteenth century, and to a large extent it codified the achievements and the current views that dominated this branch of mathematics. It was still to influence the next few decades of research. At the center of it was a well-elaborated, systematic conception of the various systems of numbers, built as a rigorous hierarchy at the basis of which lay the natural numbers, and from which all other numerical systems, up until the complex numbers, are progressively built. The subject-matter of the discipline was the study of polynomials, polynomial equations, and polynomial forms, and all relevant results and methods derived in the book directly depended upon the properties of the systems of numbers. Radical methods for solving equations obviously received a great deal of attention in this book, but so did approximation methods, which in more modern algebraic texts are left out because of their "analytical" character. Recently developed concepts such as groups and fields, and of course also the now all-important methods derived from Galois's works, were indeed treated in Weber's textbook, but only as useful tools to help dealing the main topic of polynomial equations.

At the turn of the century, then, algebra reflected a very clear conceptual hierarchy: a systematically elaborated arithmetic lying at the basis, a theory of polynomial equations built on top of it. The properties of the latter were directly derived form those of the former. Finally, a well-developed set of conceptual tools, prominently groups, affording the means to investigate this theory in the most comprehensive way.

To a large extent this picture is a very fine culmination of the long process of development that started way back in history. Fortunately it did not bring this process to a conclusion, but on the contrary, it served as a catalyzer for the new stage of development of algebra, many of whose components were already well underway in the development described in the foregoing sections.

Part III. Algebra as a discipline of structures

In 1930 a textbook of algebra was published which presented a totally new image of the discipline, one that departed fundamentally from that embodied in Weber's *Lehrbuch*. This was *Moderne Algebra* by the young Dutch mathematician Bartel Leendert van der Waerden (1903-1996), who had since 1924 attended lectures by Emmy Noether (1882-1935) in Göttingen and Emil Artin (1898-1962) in Hamburg. This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure. The main task of algebra became, under this view, the elucidation of the properties of each of these structures, and of the relationships among them. Similar questions were now asked about all these concepts, and similar concepts and techniques were used, inasmuch as possible, to deal with those questions. The classical main tasks of algebra became now ancillary. The system of real numbers, the system of rational numbers, and the system of polynomials were studied as particular instances of certain algebraic structures, and what algebra has to say about them depended on what is known about the general structures they are instances of, rather than the other way round.

Precursors of the structural approach: Hilbert, Steinitz, Noether, Artin.

Van der Waerden's book did not contain many new, individual results or concepts. The innovation lay in the new unified picture it presented of the discipline of algebra as a whole. Van der Waerden brought together, in a surprisingly illuminating manner, the results of research in algebra over the three decades following the publication of Weber's book, and in doing so, he combined the contributions of several leading German algebraists of the beginning of the century.

David Hilbert (1862-1943) was the most influential German mathematician of the turn of the century, and a leading algebraist as that. His early work on algebraic invariants reshaped this sub-discipline, through a legitimization of non-constructive proofs for the existence of certain algebraic objects (a finite basis of a system of invariants, in this case). His work on the theory of algebraic number fields in the 1890s was decisive in establishing the conceptual approach promoted by Dedekind, in opposition to the more algorithmically oriented one of Kronecker, as the dominant one in the discipline for the next decades. His work on the foundations of geometry, starting on 1899, introduced a totally new approach to the use of axiomatically defined concepts in mathematics at large. The undisputed leader of the vibrant world-class center of exact sciences in Göttingen, his influence was enormously felt through the 68 doctoral dissertations he directed, as well as through the tens of distinguished mathematicians that started their careers as students under his spell. The structural view of algebra was to a large extent the product of some of Hilbert's innovations, yet Hilbert himself basically remained a representative of the classical discipline of algebra. It is likely that the kind of algebra that developed under the influence of van der Waerden's book was of no direct appeal to Hilbert.

In 1910 Ernst Steinitz (1871-1928) published one of the most influential milestones leading to the structural image of algebra in a research piece on the abstract theory of fields. As we have seen, the concept was known from the 1880s in the works of Dedekind. Moreover, in an article of 1893 Weber had provided complete, abstract *definitions* of both groups and fields. Yet Steinitz was the first to undertake a fully abstract *research* of them. His work was highly structural in that he first established the simplest kinds of fields that any field contains (the "prime field"), and classified them according to their "characteristic". Then, he investigated how properties are passed from a field to any extension of it or to any of its sub-fields. In this way he was able to characterize all abstractly possible fields. To a great extent, van der Waerden's image of algebra may be seen as having extended to the whole discipline what Steinitz did for the more restricted domain of fields.

The greatest influence behind the consolidation of the structural image of algebra is no doubt Emmy Noether, who became the most prominent figure in Göttingen in the 1920s. Noether produced a thoroughgoing synthesis of ideas developed in earlier works of Dedekind, Hilbert, Steinitz, and others, and published a series of articles where the theory of factorization of algebraic numbers and of polynomials were masterly and succinctly subsumed under a single theory of abstract rings. She also contributed important papers to the theory of hypercomplex systems that followed a similar approach, thus definitely demonstrating the potential gains one could expect from structural research in algebra.

The last significant influence on van der Waerden's structural image of algebra to be mentioned here is that of Emil Artin, above all for his reformulation of Galois theory. Rather than speaking of the Galois group of a polynomial equation with coefficients in a field K, he focused on the group of automorphisms of it splitting field over K. Galois theory can now be seen as the study of the interrelations between the extensions of a field and the possible sub-groups of the Galois group of the original field. In this typically structural reformulation of a classical, nineteenth-century theory of algebra, the problem of solvability of equations by radicals appears as a particular application of a much general, and abstract theory.

The structural approach at work

After the late 1930s it was clear that algebra, and in particular the structural approach within it, had become a most dynamic domain of research, and its methods, results and concepts were being actively pursued by mathematicians in Germnay, France, the USA, Japan and others. It was also successfully applied to redefine several classical mathematical disciplines. Two important early examples of this are the thorough reformulation of algebraic geometry in the hands of Van der Waerden, Weil, and Oscar Zariski (1899-1986), using the concepts and the approach developed in ring theory by Emmy Noether and their successors, and the work of Marshall Stone (1903-1989), who in the late 1930s defined Boolean algebras, bringing under a purely algebraic framework ideas stemming from logic, topology and algebra itself.

Over the following decades several additional textbooks in algebra appeared following the paradigm established by van der Waerden. Prominent among these is *A Survey of Modern Algebra* first published in 1941 by Saunders Mac Lane (1909 -) and Garret Birkhoff (1921-1996), a book that became fundamental to the next several generations of the thriving algebraic research community in the USA. Algebra was increasingly taught and investigated now from a structural perspective all around the world.

Neverthles, it is important to stress that not all algebraists felt, at least at the beginning, that the new direction implied by *Moderne Algebra* was the correct one to follow. A much more classically-oriented research with deeply significant results in group theory, theory of group representations, Lie groups, etc. was still being carried out until well into the 1930s and much later. Worth of special attention in this respect are, among many others, Georg Frobenius, and Issai Schur (1875-1941), who were the most outstanding representatives of the Berlin mathematical school at the beginning of the century, and together with them, one of Hilbert's most prominent students, Hermann Weyl (1895-1955).

Algebraic superstructures: Bourbaki; Category theory

The structural approach in mathematics did not remain circumscribed to algebra alone. Very soon it became prominent in other mainstream mathematical disciplines as well, especially in the newly consolidated ones of topology and functional analysis. Some mathematicians were increasingly directing their efforts to elucidating in all these fields the properties of certain abstract constructs: topological spaces, Hilbert spaces, lattices, etc. Nevertheless, the notion of structure remained more a regulative, non-formal principle than a real mathematical concept that can be itself investigated.

It was only natural that sooner or later the question would arise of how to define structures in such a way that the concept could be fruitfully investigated. If the structural research of rings by Emmy Noether, for instance, brought new and important insights on our knowledge of the particular instances of rings previously investigated under separate frameworks (algebraic numbers, polynomials), it could perhaps be expected that a general metatheory of structures could do the same for the separate instances of, say, groups and lattices, or of algebraic structures and topological structures.

Such attempts were indeed undertaken starting in the 1940s. The first one came from a group of young French mathematicians working under the common pseudonym Nicolas Bourbaki. The founders of the group included André Weil (1906-1998), Jean Dieudonné (1906-1992), Henri Cartan (1904 -) and others. The group published a collection of textbooks, *Eléments de mathématique*, that covered several central mathematical disciplines. Over the next few decades it became extremely influential throughout the world. The subtitle of the collection, "The Elementary Structures of Analysis", is indeed revealing about the centrality that the notion of structure plays in the general of image mathematics promoted by Bourbaki. Yet, to the extent that Bourbaki's mathematics is structural, it is so in a general, non-formal way. If van der Waerden extended to all of algebra the structural approach that Steinitz introduced in the theory of fields, so can Bourbaki's *Eléments* be said to have extended this further on, into a truly broad range of hard-core mathematical disciplines.

This way of conceiving mathematics as a collection of structures at the center of which stand the three most basic ones (algebraic, topological and order-structures), and around which all the rest can be built, had yet nothing to do with a formal, mathematical definition of what a mathematical structure may be. And yet, Bourbaki did define a formal concept of structure in Book I of the collection. Interestingly, however, this concept turned out to be quite cumbersome and it was put to no real mathematical use either in the other books of the collection or in any other mathematical book or article thereafter.

The second attempt to formalize the notion of structure developed within the theory of categories and functors. The first paper on categories was published in the USA in 1942 by Saunders Mac Lane and Samuel Eilenberg (1913 - 1998), in an attempt to characterize another term then informally used in several mathematical contexts: “natural transformations”. The idea behind the categorical approach is that the essential features of any particular mathematical domain (a “category”) can be identified not by looking at the elements of an individual instance of that domain and how they behave, but rather, by looking at the way different instances within a category interconnect with each other (i.e.: by looking at the “morphisms” within a category.). Thus for instance, what characterizes the category of groups is not the fact that the elements of a group combine by means of a certain operation. Rather, the categorical approach asks about characteristic properties of group homomorphisms and in what sense they are similar to, or different from, those of, say, homeomorphissms for topological spaces. Functors, on the other hand, allow understanding interconnections across different kinds of categories: thus for instance, the discipline of algebraic topology is based on the ability to associate to each topological space certain groups that express topological properties in algebraic terms. In categorical terms this is described as functors that map each topological space (each member of the category *Top*), into a well-defined group (i.e.: a member of the category *Grp*).

Category theory did not become a universal language for all domains of mathematics but it did allow reformulations of certain mathematical disciplines for which it became the standard formulation, effectively contributing to their continued and systematic development. The two most important examples of this are algebraic topology and homology theory. In their categorical versions these disciplines were codified in two important books written by Eilenberg in collaboration with two colleagues: Norman Steenrod (1910-1917) for the first and Henri Cartan for the second. Category theory also led to new approaches in the study of the foundations of mathematics, by means of the so-called Topos theory. Some of these developments were further enhanced between 1956 and 1970 through the intensive work of Alexandre Grothendieck (1928-) and his collaborators at the IHES near

Paris, using still more general concepts based on categories, such as sheaves, motives, fibred categories and many others.

New challenges and new perspectives:

The enormous productivity of research in algebra over the second half of the twentieth century can be easily appreciated by randomly browsing through any issue of the *Mathematical Reviews*. It makes no sense to attempt to outline here the main lines of development of this kind of research. Still two main issues deserve some comment.

The first concerns the trend towards abstraction and generalization that was so prominent throughout the twenty century and was embodied in the structural approach. One should not think that this trend was exclusive. Alternative approaches have been carried out constantly, and, more importantly, the main focus of interest, of individual researchers as well as of groups of them, has moved back and forth from the general structures to the classical entities such as the real and rational numbers. This pendulum will most likely continue to swing as research goes on.

The second point to be commented upon is the introduction of new kinds of proofs and techniques that have attracted much interest but also sometimes reservations, and that are not exclusive of algebra, but that appear in this discipline in interesting ways. Three illuminating examples can illustrate this.

The first example concerns a rather classical problem: the classification of finite simple groups. A simple group is a group with no proper normal sub-groups. They were known from the time of Galois, since the reason why the general quintic is not solvable by radicals is that its Galois group is simple. A full characterization of simple groups, however, remained an open question on which group theorists worked intensively ever since. After a period of relative stagnation. a major breakthrough came in 1963 when two American group theorists Walter Feit and John G. Thomson (1932 -) proved an old conjecture by Burnside, namely, that the order of non-commutative finite simple groups is always even. Their proof was long and involved but it reinforced the belief that a full classification of finite simple groups might, after all, be possible. The completion of the task was announced in 1983 by Daniel Gorenstein (1923-1992). This classification may indeed be comprehensive, yet it is anything but clear-cut and systematic, since simple groups appear in all kind of situations and under many guises. Thus, although there is full agreement among the practitioners of the discipline that the classification has been completely achieved, there seems to be no single individual that can fully boast to know all of its details, and indeed, its full codification has turned into a collective, ongoing enterprise (much facilitated by the use of Internet techniques). This kind of “very large”, collective theorem is certainly a novel mathematical phenomenon.

A second example concerns the complex and involved question of the use of computers in proving and even formulating new theorems in algebra. It is natural that this now incipient trend will receive increased attention over the forthcoming years.

Yet a third example has to do with the introduction of probabilistic methods of proof in algebra, and in particular for solving difficult, open problems in group theory. This trend initiated in a series of papers by Paul Erdös (1913-1996) and Paul Turán (1910-1976), both of whom introduced probabilistic methods into many other branches of discrete mathematics as well. An example of an important theorem proved (by John Dixon in 1969) in this spirit states that the probability that two arbitrarily chosen elements of an alternating group of degree *n* will generate the whole group tends to 1 as *n* tends to infinity.

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Number Theory, Fields, Rings, Matrices, Vectors

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