Chapter 1
Hermann Minkowski, Relativity and the Axiomatic Approach to Physics*

Leo Corry

Abstract This article surveys the general background to Minkowski's incursion into relativity, of which Einstein's work represented just one side. Special attention is paid to the idiosyncratic, rich, and complex interaction between mathematics and physics, that stood at the center of attention of the Gttingen mathematicians since the turn of the twentieth century. In particular the article explains Minkowski's formulation of special relativity in terms of space-time against the background of David Hilbert's program for the axiomatization of physics. In addition, the article sheds light on the changing attitudes of Einstein towards mathematics, in the wake of Minkowski's work, and his increasing willingness to attribute significance to mathematical formalism in developing physical theories.

Keywords Minkowksi · Hilbert · Axiomatization · Relativity · Gravitation

1.1 Introduction

In the history of both the special and the general theories of relativity two of the leading Göttingen mathematicians of the early twentieth century play a significant role: Hermann Minkowski (1864–1909) and David Hilbert (1862–1943). Although Minkowski and Hilbert accomplished their most important achievements in pure mathematical fields, their respective contributions to relativity should in no sense be seen as merely occasional excursions into the field of theoretical physics. Minkowski and Hilbert were motivated by much more than a desire to apply their exceptional mathematical abilities opportunistically, jumping onto the bandwagon of ongoing physical research by solving mathematical problems that physicists were unable to. On the contrary, Minkowski's and Hilbert's contributions to relativity are best understood as an organic part of their overall scientific careers.

L. Corry (✉)
Cohn Institute for History and Philosophy of Science, Tel-Aviv University
e-mail: corry@post.tau.ac.il

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Indeed, a detailed examination of their careers makes it evident that a keen interest in physics was hardly ever distant from either Hilbert's or Minkowski's main focus of activity in pure mathematics.[1] Minkowski's active interest in physics dates back at least to his Bonn years (1885–1894), during which he was in close contact with Heinrich Hertz (1857–1894). In 1888 he published an article on hydrodynamics in the proceedings of the Berlin Academy [29]. From his correspondence with Hilbert, we know that during his Zürich years (1896–1902) Minkowski kept alive his interest in mathematical physics, and in particular in thermodynamics. In 1902 he moved to Göttingen, following Hilbert's strong pressure on Felix Klein (1849–1925) to create a professorship for his friend. It is well known that during his last years there, Minkowski's efforts were intensively dedicated to electrodynamics. But this was not the only field of physics to which his attention was attracted. Minkowski was commissioned to write an article on capillarity [30] for the physics volume of the *Encyclopädie der mathematischen Wissenschaften*, edited by Arnold Sommerfeld (1868–1951). At several meetings of the Göttingen Mathematical Society he lectured on this, as well as on other physical issues such as Euler's equations of hydrodynamics and Nernst's work on thermodynamics, and the evolution of the theory of radiation through the works of Lorentz, Rayleigh, W. Wien, and Planck.[2] He also taught advanced seminars on physical topics and more basic courses on continuum mechanics, and gave exercises in mechanics and heat radiation.[3]

Like Minkowski, also Hilbert developed a strong interest in physics from very early on. Throughout his career he followed the latest developments closely and taught courses and seminars on almost every current physical topic. Hilbert elaborated the principles of his axiomatic method between 1894 and 1899 as part of his current interest in problems related to the foundations of geometry, but to a considerable extent, he also reflected throughout these years on the relevance of the method for improving the current state of physical theories. Influenced by his reading of Hertz's *Principles of Mechanics*, Hilbert believed that physicists often tended to solve disagreements between existing theories and newly found facts of experience by adding new hypotheses, often without thoroughly examining whether such hypotheses accorded with the logical structure of the existing theories they were meant to improve. In many cases, he thought, this had led to problematic situations in science which could be corrected with the help of an axiomatic analysis of the kind he had masterfully performed for geometry.[4] In a course taught in Göttingen in 1905 on the logical principles of mathematics, Hilbert gave a detailed overview of how such an axiomatic analysis would proceed in the case of several specific theories, including mechanics, thermodynamics, kinetic theory of gases,

[1] For details, see [5, pp. 11–25].

[2] As registered in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (*JDMV*). See Vol. 12 (1903), 445 & 447; Vol. 15 (1906), 407; Vol. 16 (1907), 78.

[3] See the announcement of his courses in JDMV Vol. 13 (1904), 492; Vol. 16 (1907), 171; Vol. 17 (1908), 116.

[4] See [5, pp. 83–110].

electrodynamics, probabilities, insurance mathematics and psychophysics.[5] In 1905 Hilbert and Minkowski, together with other Göttingen professors, organized an advanced seminar that studied recent progress in the theories of the electron.[6] In 1907, the two conducted a joint seminar on the equations of electrodynamics.[7] In the following sections I will argue that Minkowski's work can be seen to a large extent as a particular implementation of Hilbert's program for the axiomatization of physical theories, whereby the specific, structural role of a new principle recently adopted in various physical theories – the principle of relativity – was thoroughly investigated for the first time.

Albert Einstein (1879–1955) published his famous paper on the electrodynamics of moving bodies in 1905. Minkowski read this paper at some point, as did most of his colleagues at Göttingen. We know, for instance, that in October of 1907, Minkowski wrote to Einstein asking for a reprint, in order to study it in his joint seminar with Hilbert that semester.[8] Most likely, however, a much more direct and compelling source for his keen interest in the principle of relativity and its role in physics at large stemmed directly from his reading of the famous article on the dynamics of the electron, published by Henri Poincaré (1854–1912) in January of 1906.[9] For mathematicians at Göttingen it was routine to study attentively recent work published by Poincaré in all fields of research[10] and probably Minkowski and Hilbert were in a better position than anyone else to understand the breadth and the importance of these contributions, including his 1906 article. At the same time, Minkowski did not have a high appreciation of the mathematical abilities of Einstein (who studied in his courses at Zürich). He may also have been yet unaware of the profound impact of Einstein's work on leading theoretical physicists.[11]

[5] For details on this course, see [5, pp. 138–178].

[6] See [5, pp. 127–138].

[7] A detailed list of Hilbert courses on physics appears in [5], Appendix 2.

[8] Minkowski to Einstein, October 9, 1907 (*The Collected Papers of Albert Einstein* [*CPAE*] 2, Doc. 62).

[9] [42].

[10] Thus, for instance the *JDMV* mentions reports presented at the Göttingen Mathematical Society (GMG) (some of them by Minkowski himself) on Poincaré's recent works on probability, differential equations, capillarity, mathematical physics, topology, automorphic functions, boundary-value problems, function theory, and the uniformization theorem. Cf. *JDMV* 14 (1905), 586; 15 (1906), 154–155; 17 (1907), 5.

[11] The relative interest of Minkowski and his Göttingen colleagues in Poincaré's and Einstein's respective works as possible sources of information or inspiration on the topic has, of course, nothing to do with the question of priority between these two scientists concerning the "creation of the special theory of relativity". This more general, and perhaps abstract, question, that has attracted considerable attention from historians, is rather irrelevant for our account here. For a recent discussion of this topic, that emphatically attributes priority to Poincaré and at the same times provides a rather comprehensive list of references to the existing second literature see [13] (Also available at http://albinoni.brera.unimi.it/Atti-Como-98/Giannetto.pdf). For a more recent account of Poincaré's work in relativity and its background, see [20].

Beginning in 1907, at any rate, Minkowski erected the new theory of relativity on what was to become its standard mathematical formulation, and he also devised the language in which it was further investigated. In particular, Einstein's adoption of Minkowski's formulation – after an initial unsympathetic attitude towards it – proved essential to his own attempts to generalize the theory so that it would cover gravitation and arbitrarily accelerated systems of reference. Minkowski's ideas concerning the postulate of relativity have been preserved in the manuscript and published versions of three public talks, as well as through an article posthumously published by Max Born (1882–1970), based on Minkowski's papers and on conversations between the two. The first public presentation of these ideas took place in November 5, 1907, in a talk delivered to the GMG under the name of "The Principle of Relativity,"[12] barely 1 month after requesting Einstein's paper.

Attempts to deal with the electrodynamics of moving bodies since the late nineteenth century had traditionally comprised two different perspectives: the microscopic theories of the electron and the macroscopic, or phenomenological, theories of optical and electromagnetic phenomena in moving media.[13] Whereas Einstein's 1905 relativistic kinematics concerned only Lorentz's microscopic electron theory, it was Minkowski who first addressed the problem of formulating a phenomenological relativistic electrodynamics of moving media. Thus his three public lectures on the postulate of relativity deal mainly with the macroscopic perspective, while the application of his point of view to addressing the microscopic perspective appeared in the posthumous article published by Born.

In the historiography of relativity theory, Minkowski's contributions to this domain were often judged, as were those of most of his contemporaries, against his perceived ability to understand the impact of Einstein's innovations.[14] This led to a remarkable oversight of his well-known collaboration with Hilbert as an important factor to be considered in describing and explaining his incursion into relativity theory.[15] More recent studies have adopted a broader perspective and have helped

[12] Published as [34]. For details on the printed and manuscript versions of Minkowski's work see [12, pp. 119–121]. The original typescript of this lecture was edited for publication by Sommerfeld. After comparing the published version with the original typescript, Lewis Pyenson [44, pp. 82] has remarked that Sommerfeld introduced a few changes, among them a significant one concerning the role of Einstein: "Sommerfeld was unable to resist rewriting Minkowski's judgment of Einstein's formulation of the principle of relativity. He introduced a clause inappropriately praising Einstein for having used the Michelson experiment to demonstrate that the concept of absolute space did not express a property of phenomena. Sommerfeld also suppressed Minkowski's conclusion, where Einstein was portrayed as the clarifier, but by no means as the principal expositor, of the principle of relativity." The added clause is quoted in [12, pp. 93].

[13] On the development of these two perspectives before Einstein and Minkowski, see *CPAE* 2, 503–504.

[14] Cf., e.g., [46, pp. 144]: "Hermann Minkowski, the mathematician who used Einstein's special theory of relativity to elaborate during the years 1907–1909 a theory of absolute, four-dimensional space-time ... understood little of Einstein's work and his main objective lay in imposing mathematical order on recalcitrant physical laws."

[15] For example, no such connection was considered in previous, oft-cited accounts of Minkowski's work: [12; 44; 27, pp. 238–244].

understand the immediate framework of scientific interests of Minkowski and to explain how these works fit therein, not just as a side issue to the main story of Einstein's development of the theory of relativity.[16]

In the present chapter I explain how the newly introduced relativistic ideas were combined by Minkowski with ideas embodied in Hilbert's program of axiomatization. This interpretation helps understanding the motivations and actual scope of his work and at the same time it also stresses the kind of questions that Minkowski was *not* pursuing in his work. In particular, the point of view adopted here suggests a reinterpretation of the role of Minkowski's work in the debates of the first decade of the century – much discussed in the secondary literature – concerning the ultimate nature of natural phenomena. In the earlier historiography, Minkowski's work was often presented as an attempt to elaborate and support the so-called "electromagnetic worldview" as a foundational position in physics opposed to mechanistic reductionism.[17] This debate, in which various physicists participated with varying degrees of intensity at the turn of the twentieth century, appears as irrelevant to my presentation of Minkowski's work.

1.2 The Principle of Relativity

Minkowski's first talk on electrodynamics at the meeting of the GMG in November 1907 was basically a direct continuation of his recent joint seminar with Hilbert, where they had also studied Einstein's 1905 paper. We have limited information about this seminar,[18] but we do know that in one of its meetings Hilbert discussed the electrodynamics of moving bodies. Hilbert described geometrical space as being filled with three different kinds of continua: ether, electricity and matter. The properties of these continua, he said, should be characterized by suitable differential equations. Thus the ether, a medium at rest, is characterized in terms of the magnetic and electric field intensities, $\boldsymbol{M}$ and $\boldsymbol{e}$ respectively. Electricity, a medium in motion, is characterized in terms of the current density vector and the scalar charge density, s and ρ respectively.[19] A main task of electrodynamics, Hilbert stated, is the determination of the latter two magnitudes in the presence of external forces. Hilbert seems to have expressed doubts concerning the adequacy of Lorentz's equations to describe the electrodynamics of moving bodies. At any rate, the equations discussed in the seminar were those on which Minkowski based his talk, albeit using his innovative formulation in terms of four-vectors.

[16] [55–58].

[17] See [7, Chap. 9; 19, pp. 231–242].

[18] Notes of the seminar were taken by Hermann Mierendorff, and they are preserved at the David Hilbert *Nachlass* in Göttingen (*DHN* 570/5). Cf. [44, pp. 83], for additional details.

[19] For the sake of uniformity throughout the forthcoming sections I have slightly modified the original notation and symbols. These changes are minor and should not produce interpretive problems, though. On this important point see [58].

Minkowski opened his talk by declaring that recent developments in the electromagnetic theory of light had given rise to a completely new conception of space and time, namely, as a four-dimensional, non-Euclidean manifold. Whereas physicists were still struggling with the new concepts of the theory painfully trying to find their way through the "primeval forest of obscurities," mathematicians have long possessed the concepts with which to clarify this new picture. At the center of these developments lies the principle of relativity. The impact of these developments had created a state of great conceptual confusion in many physical disciplines. The aim of Minkowski's new investigations was to clarify, to understand and to simplify the conceptual edifice of electrodynamics and mechanics, while sorting out the fundamental statements – including the principle of relativity – that lie at the basis of those disciplines. The implications derived from these first principles had to be confronted by experiment in order to validate or refute the relevant theories. Minkowski introduced here many of the mathematical concepts and terms that have come to be associated with his name and that became standard in any discussion of relativity, but he did not treat them systematically at this stage.

Minkowski was not speaking specifically about Einstein and about his 1905 paper, but rather about a broader trend that included the work of Lorentz, FitzGerald, Poincaré, and Planck. A proper elaboration of their ideas, he said, could become one of the most significant triumphs in applying mathematics to understanding the world, provided – he immediately qualified his assertion – "they actually describe the observable phenomena."[20] This latter, brief remark characterizes very aptly the nature of Minkowski's incursion into the study of the electrodynamics of moving bodies: along the lines of Hilbert's analysis of the axioms of other physical disciplines, he would attempt to understand and simplify the conceptual structures of electrodynamics and mechanics – presently in a state of great confusion, in view of the latest discoveries of physics. He would sort out the fundamental statements that lie at the basis of those structures, statements that must be confronted by experiment in order to validate or refute the relevant theories. The fundamental role played by the principle of relativity would thus be clarified.

Minkowski's main technical innovation consisted in introducing the magnitudes of four and of six components (he called the latter "*Traktoren*"), together with a matrix calculus, as the mathematical tools needed to bring to light all the symmetries underlying relativistic electrodynamics.[21] Minkowski claimed that the four-vector formulation reveals the full extent of the invariance properties characteristic of Lorentz's equations for the electron. It took a mathematician of the caliber of Minkowski to recognize the importance of Poincaré's group-theoretical interpretation of the Lorentz transformations, but he also pointed out that earlier authors, like Poincaré, had not previously emphasized that the equations satisfy this kind of purely formal property, which his newly introduced formalism made

[20] [34, pp. 927]: "falls sie tatsächlich die Erscheinungen richtig wiedergeben, ..."

[21] For the place of Minkowski's contribution in the development of the theory of tensors, see [48, pp. 168–184]. The term "four-vector" was introduced in [53].

quite evident.[22] In this earliest presentation Minkowski did not actually write down the Maxwell equations in manifestly Lorentz-covariant form. Still, he showed sketchily that if the quantities that enter the equations are written in terms of four-vectors, their invariance under any transformation that leaves invariant the expression ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$ (where $x_4 = it$) follows as a simple mathematical result. Thus formulated, the Lorentz transformations represent rotations in this four-dimensional space.

Minkowski stressed that his theory does not assume any particular worldview as part of a foundational position in physics: it treats first electrodynamics and only later mechanics, and its starting point is the assumption that the correct equations of physics are still not entirely known to us. Perhaps 1 day a reduction of the theory of matter to the theory of electricity might be possible, Minkowski said, but at this stage only one thing was clear: experimental results, especially the Michelson experiment, had shown that the concept of absolute rest corresponds to no property of the observed phenomena. He proposed to clarify this situation by assuming that the equations of electrodynamics remain invariant under the Lorentz group even *after* matter had been added to the pure field. Precisely here the principle of relativity enters the picture of physics, for Minkowski declared that this principle – i.e., invariance under Lorentz transformations – is a truly new kind of physical law: Rather than having been deduced from observations, *it is a demand we impose on yet to be found equations describing observable phenomena*.[23]

Minkowski used the four-vector formulation to show how the Galilean mechanics arises as a limiting case when $c = \infty$. Similarly, he derived the electrodynamic equations of a moving medium, making evident and stressing their invariance under the Lorentz group. He thus concluded that if the principle of relativity is to be valid also for matter in motion, then the basic laws of classical mechanics could only be approximately true. The impossibility of detecting the motion of the earth relative to the ether (following the Michelson experiment) thus implies the validity of the relativity principle.[24] As a further argument to support this rejection the classical principle of inertia Minkowski also quoted an elaborate technical argument taken from Planck's recent contribution to a relativistic thermodynamics.[25]

Minkowski concluded his lecture with a brief discussion on gravitation. Naturally, if the principle of relativity was to be truly universal it should account also for phenomena of this kind. Minkowski mentioned a similar discussion that had appeared in Poincaré's relativity article, and endorsed Poincaré's conclusion there that gravitation must propagate with the velocity of light. The purely mathematical task thus remained open, to formulate a law that complies with the relativity principle, and at the same time has the Newtonian law as its limiting case. Poincaré had indeed introduced one such law, but Minkowski regarded this law as only one among many

[22] [34, p. 929].

[23] [34, p. 931].

[24] [34, pp. 932–933].

[25] [34, pp. 935–937]. He referred to [39]. For an account of Planck's paper, see [27, pp. 360–362].

possibilities, noting that Poincaré's results had hitherto been far from conclusive. At this early stage of development of relativistic thinking in physics, the general perception was that the incorporation of Newtonian gravitation would pose only minor problems.[26] This also seems to have been Minkowski's opinion, and he left the more elaborate treatment of this point for a later occasion. Of course, he could not have imagined at this point how elusive and difficult this task would turn out to be.[27]

1.3 The Basic Equations of Electromagnetic Processes in Moving Bodies

Minkowski's second talk, "The Basic Equations of Electromagnetic Processes in Moving Bodies", was his only published text on this topic to appear before his death in 1909.[28] The talk was delivered at the meeting of the Göttingen Scientific Society (GWG) on December 21, 1907, only 2 weeks after Klein had lectured at the GMG on the possible applications of the quaternion calculus to the theory of the electron and its relation to the principle of relativity. Following Klein's lecture, Minkowski showed how the equations of electrodynamics can be simplified if the electric and magnetic magnitudes are jointly represented by means of bi-quaternions, namely, quaternions with complex components, and how this is related to the study of the significance of the principle of relativity.[29]

Minkowski's talk contained his most detailed mathematical treatment of the differential equations of electrodynamics. It also presented an illuminating conceptual analysis, very similar in spirit to Hilbert's axiomatic treatment of physical theories, of the main ideas involved in the current developments of the theories of the electron and of the role played by the principle of relativity in those theories. It is therefore not surprising that Hilbert considered this talk to be his friend's most significant contribution to electrodynamics. In his obituary of Minkowski, Hilbert stressed the importance and innovative character of the axiomatic analysis presented in that article, especially for Minkowski's derivation of the equations for moving matter starting from the so-called "World-postulate" and three additional axioms. The correct form of these equations had been theretofore a highly controversial issue among physicists, but this situation had totally changed – so Hilbert believed – thanks to Minkowski's work.[30]

[26] Cf. [36, pp. 20–21].

[27] Cf. [58] for additional details.

[28] [32].

[29] See the announcement in *JDMV* 17 (1908), 5–6.

[30] [17, pp. 93–94].

1.3.1 Three Meanings of "Relativity"

Minkowski based his conceptual analysis on a clear distinction between three possible different meanings that may be associated with the principle of relativity. First, there is the plain mathematical fact that the Maxwell equations, as formulated in Lorentz's theory of electrodynamics, are invariant under the Lorentz transformations. Minkowski called this fact the "*theorem* of relativity." Second, it seemed natural to expect, that the domain of validity of the theorem – a mathematically evident theorem, in his opinion – might be extended to cover *all* laws governing ponderable bodies, including laws that are still unknown. This is the "*postulate* of relativity," which expresses a confidence (*Zuversicht*) rather than an objective assessment concerning about the actual state of affairs. One can embrace this confidence, claim, Minkowski stressed, *without thereby committing oneself to any particular view of the ultimate relationship between electricity and matter*.[31] He compared this postulate to the principle of conservation of energy, which we assume even for forms of energy that are not yet known. Lastly, if we can assert that the expected Lorentz covariance actually holds as a relation between directly observable magnitudes relating to a moving body, then this particular relation is called the "*principle* of relativity."

From Minkowski's analysis of these three distinct interpretations of the notion of relativity we can also learn about his views on the specific contributions of the various physicists to the topics discussed. Thus, Lorentz had discovered the theorem and had also set up the postulate of relativity in the form of the contraction hypothesis. Einstein's contribution was, according to Minkowski, that of having very clearly claimed that the postulate (of relativity) is not an artificial hypothesis, but rather, that the observable phenomena force this idea upon us as part of a new conception of time. Minkowski did not mention Poincaré by name, but given the latter's conception of the general validity of the theorem, he would presumably have classified Poincaré's contribution as having also formulated the "relativity postulate." In fact, it was Poincaré who first suggested that the domain of validity of Lorentz invariance should be extended to all domains of physics. In 1904, for instance, he formulated the principle as an empirical truth, still to be confirmed or refuted by experiment, according to which the laws of physics should be the same for any two observers moving with rectilinear, uniform motion relative to each other.[32]

These attributions of his predecessors achievements served to support Minkowski's claim that his interpretation of the principle of relativity for the electrodynamics of moving bodies was a novel approach. His presentation aimed to deduce an exact formulation of the equations of moving bodies from the principle

[31] [32, pp. 353] (emphasis added).

[32] [41, p. 495; 42, p. 176]. And again in [43, p. 221]: "It is impossible to escape the impression that the Principle of Relativity is a general law of nature. . . . It is well [sic] in any case to see what are the consequences to which this point of view would lead, and then submit these consequences to the test of experiment."

of relativity, thus making clear that none of the existing formulations was fully compatible with the principle. Minkowski believed that his axiomatic interpretation of the principle of relativity was the best approach for unequivocally obtaining the correct equations. Furthermore, the invariance of these equations under the Lorentz group would follow from simple symmetry considerations.[33]

In a separate section Minkowski discussed the changes in our concepts of time implied by the introduction of the Lorentz transformations into kinematics, and in particular the impossibility of speaking about the simultaneity of two events. This section may have drawn some inspiration from a well-known article of 1906 by Kaufmann.[34] In a lengthy review of all recent experiments for testing the theories of the electron, Kaufmann established that his own results were incompatible with the "Lorentz-Einstein approach", an approach he also rejected because it did not comply with the electromagnetic world-view, which Kaufmann staunchly supported. This article attracted considerable attention, including a detailed critique by Planck, which offered open, if cautious, support for a continued study of relativity and its consequences for physics.[35] Kaufmann attributed to Einstein a new derivation of the electromagnetic equations for moving bodies in which the principle of relativity was placed at the foundation of all physical theories. In addition, he attributed to Einstein the introduction of a new conception of time that dispensed with the concept of simultaneity for two separate points in space. In his rebuttal, Planck asserted that Lorentz had introduced the principle of relativity and Einstein had formulated a much more general version of it. These two articles, which Minkowski undoubtedly read, were part of a longer series of early historical accounts that started appearing alongside the early development of the theory itself. These created different conceptions of the specific contributions of the various scientists involved.[36]

It is also noteworthy that this section appears at the end of Minkowski's discussion of the equations in empty ether. Clearly, he saw the relativity of simultaneity as a consequence of the Lorentz theorem for the equations for the ether, and thus as a fact independent of the ultimate nature of matter. The relativity of simultaneity, Minkowski moreover thought, should not pose particular difficulties to mathematicians. Familiar as the latter were with higher-dimensional manifolds and non-Euclidean geometries, they should easily adapt their concept of time to the new one. On the other hand, Minkowski noted that the task of making physical sense of the Lorentz transformations should be left to physicists, and in fact he saw the introduction of Einstein's 1905 relativity article as attempting to fulfill this task.[37]

[33] Minkowski formulated this statement in terms of four-vectors of four and six components (which he called "space-time vectors of type I and II", respectively). Vectors of type II correspond to modern second-rank, antisymmetric tensors.

[34] [21].

[35] [39]. Cf. [14, pp. 28–31].

[36] Cf. [55].

[37] [32, p. 362].

1.3.2 Axioms of Electrodynamics

Minkowski devoted a long section to analyzing in detail the Maxwell-Lorentz equations together with the underlying axioms of the theory. This section is of special interest for our purposes here, since it clearly brings to the fore the close connections between Minkowski's and Hilbert's ideas in this domain. The starting point was Lorentz's version of Maxwell's equations for the case of matter at rest in the ether, which Minkowski formulated as follows:

$$\mathrm{curl}\,\boldsymbol{m} - \frac{\partial \boldsymbol{e}}{\partial t} = \boldsymbol{s} \tag{1.1}$$

$$\mathrm{div}\,\boldsymbol{e} = \rho \tag{1.2}$$

$$\mathrm{curl}\,\boldsymbol{E} + \frac{\partial \boldsymbol{M}}{\partial t} = 0 \tag{1.3}$$

$$\mathrm{div}\,\boldsymbol{M} = 0 \tag{1.4}$$

$\boldsymbol{M}$ and $\boldsymbol{e}$ are called the magnetic and electric intensities (*Erregung*) respectively, $\boldsymbol{E}$ and $\boldsymbol{m}$ are called the electric and magnetic forces, ρ is the electric density, $\boldsymbol{s}$ is the electric current vector (*elektrischer Strom*). Further, Minkowski limited his discussion to the case of isotropic bodies by adding three conditions that characterize matter in this case:

$$\boldsymbol{e} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{M} = \mu \boldsymbol{m}, \quad \boldsymbol{s} = \sigma \boldsymbol{E}, \tag{1.5}$$

where ε is the dielectric constant, μ is the magnetic permeability, and σ is the conductivity of matter.

Minkowski sought to derive now the equations for matter in motion, and in doing so he followed and approach that strongly reminds the procedures suggested by Hilbert in his axiomatization lectures, although the details of the implementation are much more elaborated in this case than they were in any of Hilbert's presentations so far. To the equations for matter at rest Minkowski added three axioms meant to characterize the specific physical situation in mathematical terms. Thus, the three axioms are:

1. Whenever the velocity $\boldsymbol{v}$ of a particle of matter equals 0 at x, y, z, it in some reference system, then Eqs. (1.1–1.5) also represent, in that system, the relations among all the magnitudes: ρ, the vectors $\boldsymbol{s}, \boldsymbol{m}, \boldsymbol{e}, \boldsymbol{M}, \boldsymbol{E}$, and their derivatives with respect to x, y, z, it.
2. Matter always moves with a velocity which is less than the velocity of light in empty space (i.e., $|\boldsymbol{v}| = v < 1$).
3. If a Lorentz transformationacting on the variables x, y, z, it, transforms both $\boldsymbol{m}$, $-i\boldsymbol{e}$ and $\boldsymbol{M}, -i\boldsymbol{E}$ as space-time vectors of type II, and $\boldsymbol{s}, i\rho$ as a space-time vector of type I, then *it* transforms the original equations exactly into the same equations written for the transformed magnitudes.[38]

[38] [32, p. 369]. For the sake of simplicity, my formulation here is slightly different but essentially equivalent to the original one.

Minkowski called this last axiom, which expresses in a precise way the requirement of Lorentz covariance for the basic equations of the electrodynamics of moving matter, the principle of relativity. It is relevant to see in some detail how Minkowski applies the axioms to derive the equations.

Since $v < 1$ (axiom 2), Minkowski could apply a result obtained in the first part, according to which the vector **v** can be put in a one-to-one relation with the quadruple

$$w_1 = \frac{v_x}{\sqrt{1-v^2}}, \quad w_2 = \frac{v_y}{\sqrt{1-v^2}}, \quad w_3 = \frac{v_z}{\sqrt{1-v^2}}, \quad w_4 = \frac{i}{\sqrt{1-v^2}}$$

which satisfies the following relation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = -1.$$

Again from the results of the first part, it follows that this quadruple transforms as a space-time vector of type I. Minkowski called it the "velocity space-time-vector." Now, if $\mathbf{v} = 0$, by axiom 1, Eqs. (1.1–1.5) are also valid for this case. If $\mathbf{v} \neq 0$, since $|\mathbf{v}| < 1$, again the results of earlier sections allow the introduction of a transformation for which

$$w_1{}' = 0, \; w_2{}' = 0, \; w_3{}' = 0, \; w_4{}' = i.$$

In this case, we also obtain a transformed velocity $\mathbf{v}' = 0$. According to axiom 3, whatever the basic equations may be that hold for this case must remain invariant when written for the transformed variables x', y', z', t' and the transformed magnitudes $\mathbf{M}', \mathbf{e}', \mathbf{E}', \mathbf{m}', \rho', \mathbf{s}'$, and the derivatives of the latter with respect to x', y', z', t'. But, since $\mathbf{v}' = 0$, the transformed equations are (by axiom 1) just (1.1'–1.4'), obtained from (1.1–1.4) by tagging all variables. The same is true for Eq. (1.5) (although there is no need to apply axiom 3), but with ε, μ, and σ remaining unchanged. Finally, one applies the inverse of the original Lorentz transformation and, by axiom 3, it follows that the form of the basic equations for the original variables is in fact precisely that of (1.1–1.4). Minkowski thus concluded that the basic equations of electrodynamics for moving bodies are the same as the equations for stationary bodies, and the effects of the velocity of matter are manifest only through those conditions in which its characteristic constants ε, μ, and σ appear. Also, Minkowski concluded, the transformed Eq. (1.5′) can be transformed back into the original Eq. (1.5).

The arguments advanced in this section are quite different from the elaborate mathematical and physical arguments displayed throughout much of Minkowski's talk, and, at first sight, they may appear as somewhat out of place here. However, when seen in the light of the kind of axiomatic conceptual clarification promoted by Hilbert in his lectures on physics, they would seem to find a more natural place. In fact, still under the same perspective, Minkowski proceeded to check if, and to what extent, alternative, existing versions of the equations also might satisfy the principle

of relativity, as formulated in his axioms. The implicit assumption was that only equations consistent with his version of the principle of relativity could be accepted as correct. Minkowski thus found, for instance, that the macroscopic equations for moving media formulated in 1904 by Lorentz were incompatible with his principle in certain cases.[39] Likewise, the equations formulated in 1902 by Emil Cohn (1854–1944) agreed with Minkowski's own, up to terms of first order in the velocity.[40] This was a point of major significance for Minkowski. In the introduction to his article he had pointed out that, perhaps surprisingly, Lorentz's own equations for moving bodies did not correspond to the principle of relativity, and thus a major task of his article would be the formulation of the appropriate, invariant equations. In doing so, he was drawing a then unprecedented, and certainly important, distinction between Lorentz's theory of the electron and the consequences of relativity.[41] As my account here shows, this important task was reached by relying precisely on the axiomatic analysis of the theory and the principle of relativity.

1.3.3 Relativity and Mechanics

Three additional sections of this paper discuss the properties of electromagnetic processes in the presence of matter, while an appendix discusses the relations between mechanics and the postulate (not the principle!) of relativity. It is here that the similarity between Minkowski's and Hilbert's treatments of physical theories becomes most clearly manifest. Hilbert had spoken many times in the recent past about the frequent situation in the history of physics wherein new hypotheses were added to existing theories only on the basis of their intrinsic plausibility and without thoroughly checking if the former contradict the latter or any of their direct consequences. One of Hilbert's expressed aims in applying the axiomatic method to physical theories was to avoid such potential pitfalls. And indeed, it was precisely in order to avoid the danger of such a possible contradiction in the framework of the recent, exciting developments in physics that Minkowski undertook this painstaking conceptual analysis of the ideas involved. In this final section, he explored in detail the consequences of adding the postulate of relativity to the existing edifice of mechanics, as well as its compatibility with the already established principles of the discipline. The extent to which this could be successfully realized would provide a standard for assessing the status of Lorentz covariance as a truly universal postulate for all physical science.

Using the formalism developed in the earlier sections Minkowski showed that in order for the equations of motion of classical mechanicsto remain invariant under the Lorentz groupit is necessary to assume that $c = \infty$. It would be embarrass-

[39] [32, p. 372]. The article is [25].

[40] Minkowski cited here [4]. For Cohn's electrodynamics see [6, pp. 271–276].

[41] Cf. [55, footnote 15]

ing or perplexing (*verwirrend*), he said, if the laws of transformation of the basic expression

$$-x^2 - y^2 - z^2 + c^2t^2$$

into itself were to necessitate a certain finite value of c in a certain domain of physics and a different, infinite one, in a second domain. Accordingly, the postulate of relativity (i.e., our confidence in the universal validity of the theorem) compels us to see Newtonian mechanics only as a tentative approximation initially suggested by experience, which must then be corrected to make it invariant for a finite value of c. Minkowski not only thought that reformulating mechanics in this direction was possible (he asserted) in terms very similar to those found in Hilbert's lecture notes, that such a reformulation seemed to add substantially to the perfection of the axiomatic structure of mechanics.[42]

Naturally, the discussion in this section was couched in the language of space-time coordinates x, y, z, t. But Minkowski referred throughout to the properties of matter at a certain point of *space* at a given *time*, clearly separating the three elements, and focusing on the path traversed by a particle of matter throughout time. The space-time line is the collection of all the space-time points x, y, z, t associated with that particle, and the task of studying the motion of matter is defined as follows: "For every space-time point to determine the direction of the space-time line traversed by it." Likewise, the collection of all space-time lines associated with the material points of an extended body is called its space-time thread (*Raum-Zeitfaden*). One can also define the "proper time" of a given matter particle in these terms, generalizing Lorentz's concept of local time, and one can associate a positive magnitude (called *mass*) to any well-delimited portion of (three-dimensional!) space at a given time. These last two concepts lead to the definition of a rest-mass density, which Minkowski used to formulate the principle of conservation of mass. Thus, Minkowski relied here on the four-dimensional language as an effective, *formal* mathematical tool providing a very concise and symmetric means of expression, rather than as a new, *intuitive* geometrical understanding of space-time. The innovative conception usually attributed to Minkowski in this regard would only appear fully articulated in his talk of 1908 in Köln (discussed below).

Still using the same language, Minkowski analyzed the compatibility of the world-postulate with two accepted, basic principles of mechanics: Hamilton's principle and the principle of conservation of energy. He stressed with particular emphasis the full symmetry with respect to all four variables x, y, z, t, for the equations obtained. Integrating the terms of the equations of motion that had been derived by means of the Hamilton principle, he obtained four new differential equations

$$m\frac{d}{d\tau}\frac{dx}{d\tau} = R_x,$$
$$m\frac{d}{d\tau}\frac{dy}{d\tau} = R_y,$$

[42] [32, p. 393].

$$m\frac{d}{d\tau}\frac{dz}{d\tau} = R_z,$$
$$m\frac{d}{d\tau}\frac{dt}{d\tau} = R_t.$$

Here m is the constant mass of a thread, τ is the proper time, and R is a vector of type I: the *moving force* of the material points involved. The full symmetry obtained here by the adoption of the postulate of relativity struck Minkowski as highly significant, especially concerning the status of the fourth equation. Echoing once again the spirit and the rhetoric of Hilbert's lectures on axiomatization he concluded that this derivation, which he deemed surprising, entirely justifies the assertion that if the postulate of relativity is placed at the foundations of the building of mechanics, the equations of motion can be fully derived from the principle of conservation of energy alone.[43]

1.3.4 Relativity and Gravitation

Minkowski's brief treatment of gravitation follows a similar rationale: it should be proved that the World-postulate does not contradict the relevant, observable phenomena, and where necessary, the existing theory has to be suitably reformulated. Obviously, the truly universal validity of the postulate could only be asserted if it covered this domain as well, which was traditionally considered to be particularly problematic. Thus, in the closing passages, Minkowski sketched his proposal for a Lorentz-covariant theory of gravitation, much more elaborate than the one presented in his previous talk. A brief description of this section is relevant here since the general principles of the approach followed by Minkowski in developing his gravitational considerations are closely related with those of Hilbert later on. It is also noteworthy that in this section Minkowski elaborated his four-dimensional formulation even further, introducing ideas quite close to the notion of a light cone and the kind of reasoning associated with it. In this regard the overall approach of this section on gravitation can be described as much more geometric, in the basic, visual-intuitive sense of the term (albeit in four dimensions rather than the usual three), than all previous ones dealing with electrodynamics and even with mechanics.

In order to adapt Newton's theory of gravitation to the demand of Lorentz covariance Minkowski described in four-dimensional geometrical terms the force vector acting on a mass particle m at a certain point B. This vector has to be orthogonal to the world-line of the particle at B, since four-force vectors are orthogonal to four-velocity vectors. To remain close to Newton's theory, Minkowski also assumed that the magnitude of this vector is inversely proportional to the square of the distance (in ordinary space) between any two mass particles. Finally, he also assumed that the actual direction of the orthogonal vector to the world-line of m is in fact determined by the line connecting the two attracting particles. These requirements must all be

[43] [32, p. 401].

satisfied by any adaptation of Newton's laws to Lorentz covariance, but of course, Minkowski still had to be more specific in his choice of such a law. He did so in the following way: Take a fixed space-time point $B^*(x^*, y^*, z^*, t^*)$, and consider all the points $B(x, y, z, t)$ satisfying the equation

$$(x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2 = (t - t^*)^2, \quad (t - t^* \geq 0).$$

This is called the "light-structure" of B^*, and B^* is a light-point in the set of all the points located towards the concave side of the three-surface defined by the light-structure. Using the language introduced later by Minkowski himself, one can say that B^* can communicate by light signals with all points of which it is a light-point. If in the above relation, B^* is taken as variable and B as fixed, then Minkowski claimed that for an arbitrarily given space-time line there exists only one point B^* which is a light-point of B. This latter conclusion is valid only if the space-time line is (using the terminology introduced later) time-like, which is implicit in Minkowski's definition of space-time lines as world-lines of matter.[44] Given two matter points F, F^* with masses m, m^*, respectively, assume F is at space-time point B, and let BC be the infinitesimal element of the space-time line through F. This space-time line is nothing but the (modern language) word-lines of the particles at those events, with masses m, m^*. Minkowski claimed that the moving force of the mass point F at B should (*möge*) be given by a space-time vector of type I, which is normal to BC, and which equals the sum of the vector described by the formula

$$mm^* \left(\frac{OA'}{B^*D^*} \right)^3 BD^*, \tag{1.6}$$

and a second, suitable vector, parallel to B^*C^*. Figure 1.1 may help clarifying Minkowski's train of thought. The additional space-time points that appear in the

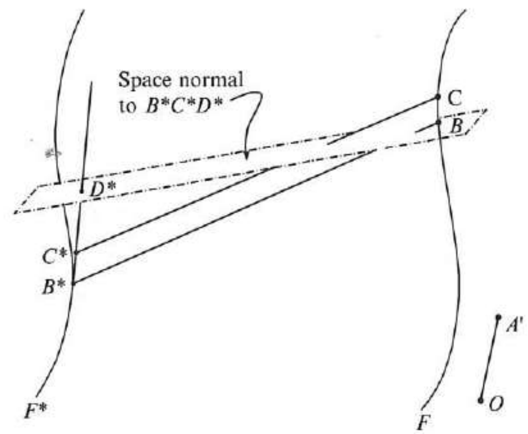

Fig. 1.1 A schematic representation of Minkowski's relativistic treatment of gravitation, for two matter points F, F^*

[44] [32, p. 393].

diagram are defined by Minkowski (without himself using any figure) as follows: B^* is the light-point of B along the space-time line of F^*; O is the origin of the coordinate system and OA' is a segment parallel to B^*C^* (C^* being the light-point along the world-line of F^*, of space-time point C) whose endpoint A' lies on the four-dimensional hyperbolic surface

$$-x^2 - y^2 - z^2 + t^2 = 1.$$

Finally, D* is the intersection point of the line through B^*C^* and the normal to OA' passing through B.

Minkowski added the assumption that the material point F^* moves uniformly, i.e., that F^* describes a straight line. Thus, at the outset he has presumably assumed that F^* moves arbitrarily. In this more general case, BC and B^*C^* represent the tangent vectors to the curves F and F^*, and they can be physically interpreted as the four-velocities of the masses with world-lines F and F^*, respectively. Now, Minkowski's gravitational force must be orthogonal to the four-velocity of F at B, and therefore orthogonal to BC. B^*C^*, on the other hand, helps to determine the distance between F and F^* in the rest-frame of the attracting body F^*, a magnitude necessary to make the gravitational law inversely proportional to it. In effect the velocity of F^* at B^* is parallel to B^*C^*, and by extending the latter into B^*D^*, Minkowski is determining the plane on which the desired distance should be measured, i.e., a plane which is normal to B^*D^* and passes through B. The space distance (not space-time) between the two points is thus given by BD^*.

Now the quantity BD^* also appears in Eq. (1.6) and in fact it gives the direction of the vector represented by the latter. But, as said above, the gravitational force should be orthogonal to BC, which is not necessarily the case for BD^*. Minkowski corrected this situation by adding to the first vector a second "suitable" one, parallel to B^*C^*. Thus the "suitable" vector that Minkowski was referring to here is one that, when added to Eq. (1.6) yields a third vector which is orthogonal to BC.

The product of the masses m and m^* appears in Eq. (1.6) and to that extent it directly corresponds to the Newtonian law. But does this equation really embody an inverse square law in the present situation? It seems that Minkowski's additional assumption, i.e., that F^* moves uniformly, could serve to answer this question (although Minkowski does not explicitly elaborate on this point). In fact, after this assumption is added, the new situation can be represented as in Fig. 1.2. If one sets the coordinates of B^* to be $(0, 0, 0, \tau^*)$, then the origin O lies on F^*. Moreover, the following values of the magnitudes involved in the equation can be deduced directly from their definitions:

$$OA' = 1; B^*D^* = t - \tau^*; (BD^*)^2 = x^2 + y^2 + z^2.$$

But B^* is a light point of B, and therefore

$$(B^*D^*)^2 = (t - \tau^*)^2 = x^2 + y^2 + z^2.$$

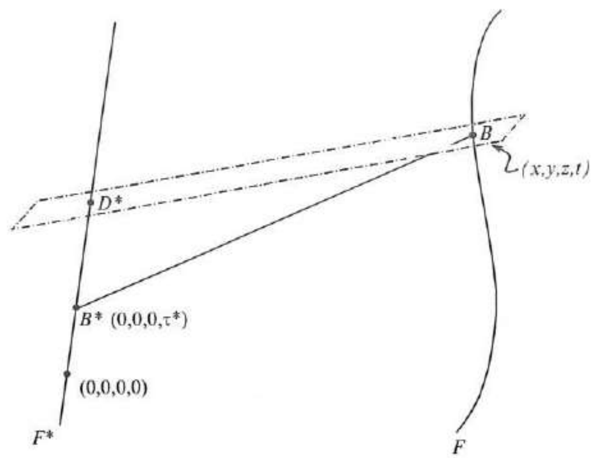

Fig. 1.2 A schematic representation of Minkowski's relativistic treatment of gravitation, with the body F^* moving uniformly

Equation (6) is thus reduced to the following:

$$mm^* \left(\frac{OA'}{B^*D^*}\right)^3 BD^* = -\frac{mm^*}{(x^2+y^2+z^2)},$$

which is the desired inverse square law of gravitation. Moreover, the assumption that F^* moves uniformly also prepares the way for Minkowski's discussion of the solar system at the end of his article (see below), by letting F^* represent the inertial motion of the sun and F the non-inertial motion of an orbiting planet.

Although many details of Minkowski's argument (such as those presented here) do not appear in the printed version of his article, all the discussion was fully conducted in the framework of space-time geometry, using only four-vectors defined on world-points and world-lines. Minkowski could thus conclude, without further comment, that the above determination of the value of the moving force is covariant with respect to the Lorentz group.

Minkowski went on to determine how the space-time thread of F behaves when the point F^* undergoes a uniform translatory motion. He asserted that starting from Eq. (1.6) as the value of the attracting force, the following four equations could be obtained:

$$\frac{d^2x}{d\tau^2} = -\frac{m^*x}{(t-\tau^*)^3}, \quad \frac{d^2y}{d\tau^2} = -\frac{m^*y}{(t-\tau^*)^3}, \quad \frac{d^2z}{d\tau^2} = -\frac{m^*z}{(t-\tau^*)^3}, \tag{1.7}$$

and

$$\frac{d^2t}{d\tau^2} = -\frac{m^*x}{(t-\tau^*)^2}\frac{d(t-\tau^*)}{dt}. \tag{1.8}$$

Since the relation $x^2+y^2+z^2 = (t-\tau^*)^2$ holds true, Eq. (1.7) is a set of equations similar to the motion equations of a material point under the Newtonian attraction of

a fixed center, as Minkowski stated, substituting instead of the time t the proper time τ of the particle. On the other hand, Eq. (1.8) establishes the dependence between the proper time of the particle and the time t. Using these equations, Minkowski added some brief calculations concerning the orbits and expected revolution times of planets and inferred – using the known values of the mass of the Sun as m^* and of the axis of the Earth's orbit – that his formulas yielded values for the eccentric anomalies in the planetary orbits of the order of 10^{-8}. He concluded with two remarks: first, that the kind of attraction law derived here and the assumption of the postulate of relativity together imply that gravitation propagates with the velocity of light. Second, that considering the small value obtained above for Kepler's equation for eccentric anomalies, the known astronomical data cannot be used to challenge the validity of the laws of motion and modified mechanics proposed here and to support Newtonian mechanics.[45]

Minkowski's treatment of gravitation was extremely sketchy and tentative. An attentive reading of it raises more questions that it seems to answer. Some of these questions have been formulated in the foregoing paragraphs, but more can be added. For instance: Is Minkowski's gravitational force in any sense symmetric with respect to F and F^*? What kind of conservation laws arise within such a theory? Minkowski did not address these issues, either in the article or elsewhere. Rather than addressing the issue of gravitation in detail, when writing this article Minkowski's main concern was clearly to investigate the logical status of the principle of relativity as applied to all physical domains and the plausibility of assuming that it must also hold when dealing with gravitation.

Still, the theory outlined in this lecture was, together with Poincaré's, the starting point of the attempts to extend the validity of the principle of relativity to cover gravitation as well. Einstein himself addressed the same task in an article submitted for publication on December 4, 1907, less than 3 weeks before Minkowskii's talk, in which he raised for the first time the question whether the principle of relativity could be extended to cover accelerated, rather than only inertial reference systems.[46] Although Einstein formulated here for the first time what he later called the principle of equivalence – a fundamental principle of his general theory of relativity – his 1907 attempt did not directly lead to an extension of the validity of relativity. Einstein did not return to this topic until 1911, when his actual efforts to generalize relativity really began. In his 1907 paper Einstein mentioned neither Minkowski nor Poincaré. Nor did Minkowski mention this article of Einstein, and one wonders if at this point he had already read it. Minkowski's approach to electrodynamics and the principle of relativity came to provide the standard language for future investigations, but his specific argumentation on gravitation attracted little if any attention. Minkowski himself mentioned the issue of gravitation once again in his next article, "Space and Time," but only in passing. Arnold Sommerfeld, whose 1910 article contributed more than any other work to systematize and disseminate

[45] [32, p. 404].

[46] [8].

Minkowski's four- and six-vector formalism, claimed that Minkowski's approach to gravitation was no better than Poincaré's, and that if they differed in any respect – as Minkowski had claimed in his article – it was in their methods rather than in their results.[47] Unfortunately, we do not know how Minkowski would have reacted to Sommerfeld's interpretation on this point.

I summarize this section by stressing that Minkowski sought to investigate, in axiomatic terms, the conceptual consequences of applying the postulate of relativity in domains other than electrodynamics. In this framework he addressed, besides mechanics, gravitation and showed how an argument could be worked out for the claim that there was no *prima facie* reason to assume that the postulate of relativity contradicts the observable effects of phenomena pertaining to this latter domain. He concluded that one could envisage the possibility of a truly articulate Lorentz-covarianttheory of gravitation which would approximate the Newtonian theory as a limiting case. It seems, however, that neither Minkowski nor Hilbert considered this theory as anything more than a very preliminary attempt. On the other hand, this whole lecture, and especially its final sections, helps clarifying the kind of motivations underlying Minkowski's investigation of the place of the principle of relativity in physics. Moreover, this particular talk of 1907 shows very clearly how the geometric element ("geometric" taken here in its intuitive-synthetic, rather than in its formal-analytical, sense) entered Minkowski's treatment only gradually, and that an immediate visualization, in geometric terms, of the consequences of the adoption of the principle of relativity in mechanics was not an initial, major motivation behind his attempt. Such a geometrical elements becomes central only in his next text on electrodynamics, "Space and Time".

1.4 Space and Time

Minkowski first presented his views on relativity outside Göttingen on September 21, 1908, when he delivered a lecture at the annual meeting of the German Society of Natural Scientists and Physicians in Köln. The text of his lecture was later published as "*Raum und Zeit*", Minkowski's best known contribution to the special theory of relativity and to the new conception of space and time associated with it. Both the opening and the closing passages of the text have repeatedly been quoted as encapsulating the essence of Minkowski's views. The opening passage of the talk was a rather dramatic proclamation:

> Gentlemen! The conceptions of space and time which I would like to develop before you arise form the soil of experimental physics. Therein lies their strength. Their tendency is radical. Henceforth, space by itself, and time by itself, are doomed to fade away in the shadows, and only a kind of union of the two will preserve an independent reality.[48]

[47] [53, p. 687]. On pp. 684–689 one finds a somewhat detailed account of the physical meaning of Minkowski's sketch for a theory of gravitation, and a comparison of it with Poincaré's.

[48] [33, p. 431].

In the closing passage he concluded: "The validity without exception of the world-postulate, I would like to think, is the true nucleus of an electromagnetic image of the world, which, discovered by Lorentz, and further revealed by Einstein, now lies open in the full light of day." These two passages have helped consolidate the image of Minkowski's geometrically motivated approach to relativity and of his alleged commitment to the electromagnetic view of nature. Still, an analysis of his text against the background of Hilbert's program for the axiomatization of physical theories, and in the spirit of the previous two sections, makes clear that such a commitment did not exist, and at the sane allows interpreting these passages in a different way, as will be seen now.

Minkowski started by presenting two kinds of invariance that arise in connection with the equations of Newtonian mechanics. First, the invariance associated with an arbitrary change of position, and second, the one associated with uniform translation. Our choice of a particular point as $t = 0$ does not affect the form of the equations. Although these two kinds of invariance can both be expressed in terms of the groups of invariance they define with respect to the differential equations of mechanics, traditional attitudes towards these respective groups had been utterly different. For, whereas the existence of the group corresponding to the first invariance had usually been seen as expressing a fundamental property of space, the existence of the second (i.e., the group of Galilean transformations) had never attracted any special interest as such. At best, Minkowski said, it had been accepted with disdain (*Verachtung*) in order to be able to make physical sense of the fact that observable phenomena do not enable one to decide whether space, which is assumed to be at rest, is not after all in a state of uniform translation. It is for this reason, Minkowski concluded, that the two groups carry on separate lives with no one thinking to combine them into a single entity.

Minkowski thought that this separation had a counterpart in the way the axiomatic analysis of these two scientific domains had typically been undertaken: in the axiomatization of mechanics, the axioms of geometry are usually taken for granted, and therefore the latter and the former are never analyzed simultaneously, as part of one and the same task.[49] We know precisely what Minkowski meant by this latter assertion. For in his 1905 lectures on the axiomatization of physics, Hilbert had discussed the axiomatization of the laws of motion by adding to the already accepted axioms of geometry separate axioms meant to define time through its two basic properties, namely, its uniform passage and its unidimensionality (*ihr gleichmäßiger Verlauf und ihre Eindimensionalität*).[50] This traditional separation of mechanics and geometry was more explicitly manifest in relation with their respective invariance groups, as explained above, but it had also been implied in the way their axiomatic definitions had been introduced. Minkowski's brilliant idea in this context was to put an end to this separation and to combine the two invariance

[49] [33, p. 431]: "Man ist gewohnt, die Axiome der Geometrie als erledigt anzusehen, wenn man sich reif für die Axiome der Mechanik fühlt, und deshalb werden jene zwei Invarianten wohl selten in einem Atmenzuge genannt." The standard English translation of Minkowski's lecture [35] is somewhat misleading here, as in many other passages.

[50] [16, p. 129]. For details, see [5, pp. 138–153].

groups together. He assumed that this combination would lead to a better understanding of the reality of space and time, and of the laws of physics. The aim of his talk was to explain the implications of such a move.

Minkowski's audience was mainly composed of natural scientists rather than mathematicians. This certainly influenced the kinds of arguments he used and the emphases he chose to adopt. In particular, he stressed from the outset that the ideas presented in the lecture were independent of any particular conception of the ultimate nature of physical phenomena. As in his two previous lectures on the same topic, Minkowski intended his arguments to be an exploration of the logical consequences of adopting the postulate of relativity in the various domains of physics, without necessarily committing himself to any particular view. Therefore, he put forward his arguments in a way intended to prevent any physicist, whatever his basic conception of physical phenomena, from reacting to these ideas with *a priori* suspicion or hostility. Thus, Minkowski's arguments were meant to be compatible with any possible belief concerning the ultimate nature of mass, electromagnetic processes and the ether, and the relationships among these: "In order not to leave a yawning void anywhere," he said, "we want to imagine, that at any place in space at any time something perceptible exists. In order not to say matter or electricity, I will use the word 'substance' to denote this something."[51] *Substance* was therefore a general category rather than being bound to a particular physical interpretation of mass, ether, electricity or any other candidate. In a later passage in which he referred to the velocity of light in empty space, he exercised the same kind of caution: "To avoid speaking either of space or of emptiness, we may define this magnitude in another way, as the ratio of the electromagnetic to the electrostatic unit of electricity."[52]

Assuming that we are able to recognize a substantial point as it moves from a first four-coordinate "world-point," to a second one, Minkowski declared in the introduction that the world can be resolved into world-lines, namely, collections of all the world-points associated with a substantial point when t takes all values between $-\infty$ and ∞. He added that the laws of physics attain their most perfect expression when formulated as relations between such world-lines.

1.4.1 Groups of Transformations

In his first talk on the principle of relativity in 1907, Minkowski had already shown that the assumption of the principle of inertia implies that the velocity of propagation of light in empty space is infinite. This time he discussed this implication, while focusing on certain formal properties of the groups defined by the Galilean transformations and by the Lorentz transformations. The first group expresses the fact that if the x, y, z axes are rotated around the origin of coordinates while $t = 0$, then

[51] [33, p. 432].

[52] [33, p. 434; 35, p. 79].

the expression $x^2 + y^2 + z^2$ remains invariant. The second group expresses the fact that the laws of mechanics remain unchanged under the transformations that send x, y, z, t to $x - \alpha t, y - \beta t, z - \gamma t, t$, with any constant coefficients α, β, γ. Under these transformations, the t-axis can be given whatever upward direction we choose. But how is the demand of orthogonality in space, asked Minkowski, related to this complete freedom of the t-axis? To answer this question Minkowski suggested that one must consider four-dimensional space-time and a more general kind of transformation, namely, those that leave invariant the expression $c^2t^2 - x^2 - y^2 - z^2 = 1$. These transformations turn out to depend on the value of the parameter c and thus classical mechanics appears as a special case of a more general class of theories. He stressed the geometrically intuitive elements of his arguments, by focusing on the case $c^2t^2 - x^2 = 1$, which is graphically represented as a hyperbola on the plane x, t (Fig. 1.3):

Here OB is the asymptote ($ct - x = 0$), and the orthogonal segments OC and OA have the values $OC = 1$ and $OA = 1/c$. Choose now any point A' on the hyperboloid, draw the tangent $A'B'$ to the hyperbola at A', and complete the parallelogram $OA'B'C'$. If OA' and OC' are taken as new axes, x', t' respectively, and we set $OC' = 1, OA' = 1/c$, then the expression for the hyperbola in the new coordinates retains its original form $c^2t'^2 - x'^2 = 1$. Hence, OA' and OC' can now be defined as being themselves orthogonal and thus the hyperbola construction helps to conceive orthogonality in a way that departs from the usual Euclidean intuition. The parameter c determines in this way a family of transformations that, together with the rotations of space-time around the origins of coordinates, form a group, the group G_c. But then – again from geometric considerations – one sees that when c grows infinitely large, the hyperbola approximates the x-axis and, in the limit case, t' can be given any upward direction whatever, while x' approaches x indefinitely. This geometrical argument thus shows that G_∞ is nothing but the above described group of transformations G_c associated with Newtonianmechanics.

This illuminating connection between the two main groups of transformations that arise in physics allowed Minkowski to digress again and comment on the relation between mathematics and physics:

> This being so, and since G_c is mathematically more intelligible than G_∞, it looks as though the thought might have struck some mathematician, fancy-free, that after all, as a matter of fact, natural phenomena do not possess an invariance with the group G_∞, but rather with a

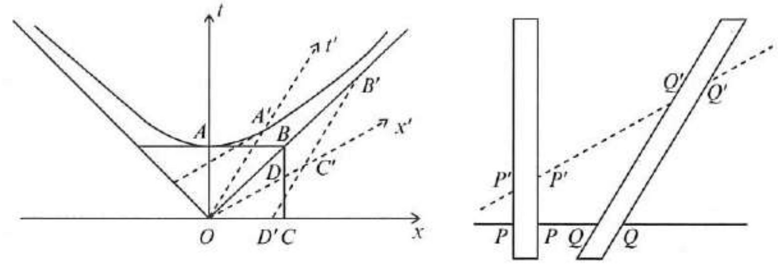

Fig. 1.3 Original diagram of Minkowski's "Space and Time" talk

group G_c, c being finite and determinate, but in ordinary units of measure, *extremely great*. Such a premonition would have been an extraordinary triumph for pure mathematics. Well, mathematics, though it can now display only staircase-wit, has the satisfaction of being wise after the event, and is able, thanks to its happy antecedents, with its senses sharpened by an unhampered outlook to far horizons, to grasp forthwith the far-reaching consequences of such a metamorphosis of our concept of mature. [33, p. 434; 35, p. 79])

It is not evident, on first reading, what Minkowski meant here when he said that G_c is "mathematically more intelligible" than G_∞, but apparently he was pointing to the fact that the group of Galilean transformations, which in itself had failed to attract any interest from mathematicians, becomes much more mathematically interesting when seen in the more general context of which it appears as a limiting case. In retrospect, Minkowski concluded, this situation might seem to suggest that mathematical insight could have sufficed to realize what is involved here, but in fact this was not the case, and physical considerations were necessary.

The invariance under the group G_c of the laws of physics in a four-dimensional space-time has for Minkowski an additional, important consequence that reinforces – from a different perspective and in a much more compelling fashion – a point of view earlier elaborated in Hilbert's writings, namely, the view of geometry (i.e., the science of sensorial space) as a natural science on which all other physical sciences are grounded. Yet, what Hilbert had initially expressed as an epistemologically grounded conception, and had later developed when discussing the axioms of mechanics on the basis of the axioms of geometry, appears here in the opposite direction: the latest developments of physical science have raised the need to reconsider our basic conception of space and time in such a way as to recognize that geometry is essentially embedded in physics. Thus, to conclude this section of his lecture Minkowski said:

In correspondence with the figure described above, we may also designate time t', but then must of necessity, in connection therewith, define space by the manifold of the three parameters x', y, z, in which case physical laws would be expressed in exactly the same way by means of x', y, z, t', as by means of x, y, z, t. We should then have in the world no longer space, but an infinite number of spaces, analogously as there are in three-dimensional space an infinite number of planes. Three dimensional geometry becomes a chapter in four-dimensional physics. (ibid.)

1.4.2 Empirical Considerations

So much for the formal, geometrical considerations, but of course the question arises: what empirical facts compel us to adopt this new conception of space? Moreover, can we be sure that this conception never contradicts experience? Is it useful in describing natural phenomena? These questions were discussed by Minkowski in the following three sections of his talk. First, he observed that by means of a suitable transformation the substance associated with a particular world-point could always be conceived as being at rest. This he considered to be a fundamental axiom of his theory of space-time. A direct consequence of the axiom is that every possible velocity in nature is smaller than c. In his second 1907 lecture Minkowski

had taken this consequence in itself as a central axiom of the electrodynamics of moving bodies. Formulated in these terms, he felt, it had a somewhat "unpleasant" appearance that raised mistrust, but in the present four-dimensional formulation it could be grasped more easily.

Using the groups G_c and G_∞, Minkowski explained the problems raised by the Michelson experiment, given the different invariance groups characteristic of different physical disciplines. He stressed that the concept of a rigid body may have a coherent meaning only in a mechanics based on the group G_∞, and that the contraction hypothesis had been introduced by Lorentz in order to account for the divergence detected between theory and experiment. Remarkably enough, in spite of having stressed pompously in the opening passage of his talk that the origin of these new conceptions was fully rooted in experiment, this is the only reference in the whole text to anything of the sort. In fact, Minkowski preferred to ignore recent results by Kaufmann already mentioned above, that allegedly refuted the theory of relativity.[53] Admitting that the contraction hypothesis in its original form "sounds extremely fantastical," he proceeded to show that it is entirely coherent when seen in terms of] the new conception of space and time, and that the latter clarified the former completely. Minkowski's explanation was fully geometrical and it relied on a straightforward verification of the properties of a rectangle and a parallelogram drawn on the two-dimensional figure introduced in the first section. At this point Minkowski also characterized Einstein's contribution in this context, as explaining the nature of local time. Whereas Lorentz had introduced the concept as a tool for better understanding the contraction hypothesis, Einstein "clearly recognized that the time of the one electron is just as good as that of the other."[54] Thus, Minkowski saw that Einstein had essentially undermined the idea of *time* as a concept unequivocally determined by phenomena. But then, in spite of the importance of this achievement, neither Einstein nor Lorentz undertook a similar attack on the concept of *space*. Minkowski considered such an attack to be indispensable in uncovering the full implications of the postulate of relativity, and he saw his own ideas as having contributed to the full achievement of that aim. It was in this framework that he introduced the term "World-postulate" instead of relativity:

When [the attack on the traditional concept of space] has been undertaken, the word *relativity-postulate* for the requirement of invariance with the group G_c seems to me very feeble. Since the postulate comes to mean that spatio-temporal phenomena manifest themselves only in terms of the four-dimensional world, but the projection in space and in time may still be performed with certain liberty, I prefer to call it the *postulate of the absolute world* (or briefly, the world-postulate). [33, p. 437][55]

[53] This point has been raised by Scott Walter [56, p. 52] in his perceptive study of the rhetoric strategy followed by Minkowski, the mathematician, in addressing a public of non-mathematicians.

[54] [33, p. 437 ([35, p. 83])]. In his obituary of Minkowski, Hilbert [17, p. 90] repeated this assessment. For a discussion of the differences in the conception of time in Einstein's and in Minkowski's theories, see [56], § 3.5.

[55] Minkowski's original sentence – "noch mit einer gewissen Freiheit vorgenommen werden kann, ..." – appears in the English translation [35, p. 83] as: "may still be undertaken with a certain degree of freedom." This rendering seems to me somewhat misleading in this context.

It is significant that in this talk Einstein's work becomes a much more important focus of reference for Minkowski than in the previous two, particularly Einstein's innovative conception of time. It is very likely that by this time Minkowski had already read Einstein's 1907 article mentioned above. This survey article had been written at the request of Johannes Stark (1874–1957), editor of the *Jahrbuch der Radioaktivität und Elektronik*, following the recent publication of Kaufmann's criticism of relativity. Attempting to strengthen the theoretical and experimental support for his theory, Einstein now stressed the *similarities* between Lorentz's and his own work. He presented the latter as genetically related to the former (and, implicitly, also superior to it) rather than presenting these as two alternative approaches to the same problem. At the same time he explicitly attributed a central place to the Michelson-Morley experiment in the development of the whole theory (and implicitly in the development of his own).[56] Einstein himself considered this presentation of his theory to be simpler and more intuitive than the one of 1905 where he had striven, above all, for "unity of presentation".[57] The rhetoric of Minkowski's talk connects smoothly and in visible ways with the spirit and contents of Einstein's 1907 article.

In the third part of the lecture, Minkowski showed that the world-postulate provides a much clearer understanding of the laws of physics, by allowing a symmetrical treatment of the four coordinates x, y, z, t. In this first section he introduced the concept – only implicit in his earlier lectures – of a light-cone (in fact, he only spoke separately of the front- and back-cones of a point O) and explored its usefulness, especially in dealing with the concept of acceleration.

1.4.3 Relativity and Existing Physical Theories

In the last two sections, Minkowski addressed again the main point discussed in his previous talk, namely, the compatibility of the principle of relativity with existing physical theories, or, as he put it here, that "the assumption of the group G_c for the laws of physics never leads to a contradiction." In order to show this, Minkowski understood that it was "unavoidable to undertake a revision of the whole of physics on the basis of this assumption." Such a revision had in fact already begun. Minkowski cited again Planck's recent article on thermodynamics and heat radiation,[58] as well as his own earlier lecture, already published by then, where the compatibility of the postulate of relativity with the equations of electrodynamics and of mechanics (retaining, he stressed, the concept of mass) had been addressed. With reference to the latter domain, Minkowski elaborated this time on the question of how the

[56] Cf. [55, pp. 275–281]. For debates on the actual role of the Michelson-Morley in the development of Einstein's ideas and its historiography, see [15; 18, pp. 279–370; 54]. At any rate, Einstein had read about the experiment as early as 1899. Cf. *CPAE* 1, Doc. 45, 216.

[57] Einstein to Stark, November 1, 1907 (*CPAE* 5, Doc. 63).

[58] [40]. Another remarkable aspect in the rhetoric of Minkowski in this talk is the total absence of references to Poincaré. On possible reasons for this, see [56, pp. 60–62].

expressions of force and energy change when the frame of reference changes. He then showed how the effects produced by a moving point-charge, and in particular the expression of its ponderomotive force, can be best understood in terms of the world postulate. He stressed the simplicity of his own formulation as compared with what he considered the cumbersome appearance of previous ones.

Finally, in a brief passage, Minkowski addressed the question of gravitation, noting that the adoption of the world-postulate for mechanics as well as for electrodynamics eliminated the "disturbing lack of harmony" between these two domains. Referring back to his published lecture of 1907, he asserted that, by introducing in the equations of motion under gravitation the proper time of one of the two attracting bodies (which is assumed to be moving, while the other is at rest), one would obtain a very good approximation to Kepler's laws. From this he concluded, once again, that it is possible to reformulate gravitation so as to comply with the world-postulate.

In his closing remarks, Minkowski addressed the question of the electromagnetic world-view and the postulate of relativity, which he had expressly bypassed throughout the lecture. For Minkowski, it was not the case that all these physical domains were compatible with the world-postulate (merely) because their equations had been derived in a particular way; the postulate had a much more general validity than that. It is in this light that we must understand the often-quoted closing passage of the lecture. The equations that describe electromagnetic processes in ponderable bodies completely comply with the world-postulate, Minkowski remarked. Moreover, as he intended to show on a different occasion, in order to verify this fact it is not even necessary to abandon Lorentz's erudite (*gelehrte*) derivation of these fundamental equations, based on the basic conceptions (*Vorstellungen*) of the theory of the electron.[59] In other words, whatever the ultimate nature of physical processes may be, the world-postulate, i.e., the universal demand for invariance under the group G_c of the equations expressing the laws of physical processes, must hold valid. This is what we have learnt from the latest developments in physics and this is what Minkowski expressed in his well-known assertion:

> The validity without exception of the world-postulate, I like to think, is the true nucleus of an electromagnetic image of the world, which, discovered by Lorentz, and further revealed by Einstein, now lies open in the full light of day. In the development of its mathematical consequences there will be ample suggestions for experimental verification of the postulate, which will suffice to conciliate even those to whom the abandonment of the old-established views is unsympathetic or painful, by the idea of a pre-established harmony between mathematics and physics [33, p. 444; 35, p. 91].

Clearly, then, in reading this passage we need not assume that Minkowski was trying to advance the view that all physical phenomena, and in particular the inertial properties of mass, can be reduced to electromagnetic phenomena. Nor is it necessary to determine to what extent Minkowski had understood Einstein's innovative point of view in his paper on the electrodynamics of moving bodies, as compared to all

[59] [33, p. 444]. Also here the translation [35, pp. 90–91] fails to convey the meaning of the original passage.

the other sources from which his theory took inspiration. Rather, Minkowski only claimed here that the electromagnetic world-view is nothing but what the world-postulate asserts: the belief in the general validity of the world-postulate is all that there is, and can be, to the electromagnetic world-view. A similar attitude was found in Hilbert's 1905 lectures on physics, when he analyzed in axiomatic terms the basic assumptions of a theory that are necessary for the derivation of its main theorems, but avoided, as much as possible, any commitment to a particular world-view. Both Minkowski and Hilbert believed that in constructing the mathematical skeleton of all physical theories, certain universal principles must be postulated (the world-postulate and general covariance, but also the energy principle and the continuity principle); even in the face of new empirical discoveries that will force changes in the details of individual theories, these general principles will continue to hold true. Moreover, the idea of a pre-established harmony of mathematics and physics, so popular in the discourse of the Göttingen scientific community, can be traced back to the belief in the existence of such universal principles, rather than to the specific contents of particular, probably provisional, physical theories expressed in mathematical terms. The idea of a "true nucleus" (*der wahre Kern*) of physical theories that is preserved amidst other, presumably more cosmetic traits, will also resurface in remarkable circumstances in the work of Hilbert on general relativity.[60]

1.5 Max Born, Relativity, and the Theories of the Electron

In "Time and Space", Minkowski had set to verify the universal validity of the postulate of relativity at the macroscopic level. In the closing passages of the lecture he declared that on a future occasion he intended to do so at the microscopic level as well, namely, starting from Lorentz's equations for the motion of the electron. On July 28, 1908, he gave a talk at the meeting of the Göttingen Mathematical Society on the basic equations of electrodynamics. Although no complete manuscript of this lecture is known, a very short account, published in the *JDMV* seems to indicate that Minkowski addressed precisely the microscopic derivation of the equations using the principle of relativity.[61] Be that as it may, he was not able to publish any of these ideas before his untimely death on January 12, 1909. We nevertheless have a fair idea of what these ideas were, from an article published by Max Born in 1910, explicitly giving credit for its contents to Minkowski. Born used Minkowski's unfinished manuscripts and the ideas he heard in the intense conversations held between the two before Minkowski's death.

According to Born's introduction, the starting point of Minkowski's "*Grundgleichungen*" had been the assumption of the validity of the Maxwell equations for stationary bodies, inductively inferred from experience. This point of view,

[60] See [5, pp. 399–403].

[61] *JDMV* 17 (1909), 111.

explained Born, differed from Lorentz's, which accounted for processes in material bodies in terms of certain hypotheses about the behavior of the electrons that compose those bodies. Lorentz had considered three kinds of electrons. First, there were conduction electrons (*Leitungselektron*), whose movement is independent of matter and whose charge constitutes "true electricity." Second, polarization electrons provided a state of equilibrium inside molecules of matter; these electrons, however, can be dislocated from this state through the action of the electromagnetic field. The variable electricity density produced in this way is known as the "free electricity." Third came the magnetization electrons that orbited around central points inside matter, thus giving rise to magnetic phenomena. Lorentz's equations for electromagnetic processes in material bodies were based on the mean values of the magnitudes of the convection current due to the three types of electron. Yet as Minkowski had shown in his "*Grundgleichungen*", in certain cases – specifically, in the case of magnetized matter – the equations thus obtained contradict the postulate of relativity.

The specific aim of the article, then, was to extend the validity of the postulate to cover all cases, including the problematic one pointed out by Minkowski in his earlier article. But for all the assumptions concerning the complex structure of matter that the above discussion implies, Born understood the need to stress, as Minkowski had done before him, the *independence* of this study from a particular conception of the ultimate nature of matter, ether or electricity. He thus explained that "among the characteristic hypotheses of the electron theory, the atomic structure of electricity plays only a limited role in Lorentz's derivation of the equations," given the fact that mean values have been taken over "infinitely small physical domains", so that all this structure is completely blurred, and the mean values, in the final account, appear as continuous functions of time and location. Born thus justified his adoption of Lorentz's approach to the derivation of the equations, without thereby committing himself to any ontological assumptions. He declared very explicitly:

> We hence altogether forgo an understanding of the fine structure of electricity. From among Lorentz's conceptions, we adopt only the assumptions *that electricity is a continuum that pervades all matter, that the former partially moves freely inside the latter and partially is tied to it, being able to carry out only very reduced motions relative to it.*

If we want to come as close as possible to Lorentz, then all the magnitudes introduced below should be considered as Lorentzian mean values. It is however not necessary to differentiate among them, using special symbols, as if they were related to the various kinds of electrons, since we never make use of the latter.[62]

Following Minkowski's death, Born went on to develop his own ideas on relativity, which he had begun to consider following his reading of Einstein. A fundamental contribution of Born was the introduction of the Lorentz-invariant concept of a rigid body, a concept to which Born was led while working on the problem of the self-energy of the electron. As we saw above, Minkowski had already made it clear in

[62] Minkowski 1910, 61 (Italics in the original).

"Space and Time" that the traditional concept of rigid body did not make sense outside Newtonian mechanics. Born's interest in this question implied an involvement in the Abraham-Lorentz debate concerning the independence or dependence of the mass of the (rigid or deformable) electron on its velocity, and, in the question of the possible electromagnetic nature of the mass of the electron. In his autobiography, Born mentions that in their discussions of these issues, Minkowski "had not been enthusiastic about Born's own ideas but had raised no objections."[63] One wonders whether Minkowski's lack of enthusiasm was not perhaps connected to Born's particular interest in the electromagnetic mass of the electron, a topic which Minkowski persistently tried to avoid in his own work.

Both Abraham and Lorentz had calculated the self-energy of a charged, rigid body moving uniformly and used this energy as the Hamiltonian function for deriving the equations of motion. Born doubted the validity of an additional assumption implicit in their calculations, namely, that the energy calculated for uniform motion is the same for accelerated motion, since in an accelerated body different points have different velocities and therefore, according to the principle of relativity, different contractions. The classical concept of a rigid body is thus no longer applicable. Without entering to all the technical details of Born's derivation, I will nevertheless mention that his definition is based on finding a Lorentz-covariant expression of the distance between any two space-time points; the classical distance between two points in a body is given by

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2,$$

which is clearly not Lorentz-covariant.[64]

Born discussed the Lorentz-covariant definition of rigidity in two articles published in 1909. In the first, submitted on January 9 (just 3 days before Minkowski's death), he discussed the relation between the concept of mass and the principle of relativity. This article still reflects the direct influence of Minkowski's point of view. Born referred in the introduction of this article to the "Abraham-Sommerfeld theory of the rigid electron", whose main task he described as that of reducing the inertial mass of the electron to purely electrodynamic processes. The theory, however, does not satisfy the "Lorentz-Einstein principle of relativity." On the other hand, said Born, the latter principle has not led to a satisfactory explanation of inertial mass. The equations of motion formulated by Lorentz, Einstein and Minkowski are suggestive approximations of the Newtonian equations, which at the same time satisfy the relativity principle of electrodynamics. The concept of mass is thus modified in the works of the three so as to fit that principle without, however, explaining the concept in electrodynamical terms.

Born's treatment of mass was intended as an analogy to Minkowski's ideas, but applied in the framework of the Abraham-Sommerfeld theory. Minkowski had

[63] [3, p. 132].

[64] For a more detailed discussion of Born's concept of rigid body and its impact, see [26; 27, pp. 243–257].

modified the Hamiltonian principle of classical mechanics so as to make the ensuing equations of motion fit the relativity principle. The variational equation to which this principle gives raise yields two integrals, one of which expresses the effect of the mass. Born intended to introduce a similar generalized Hamiltonian involving only electromagnetic magnitudes, and to derive the mass in a way similar to Minkowski's. However, it is noteworthy that for all of his interest in the Abraham-Sommerfeld theory, Born took pains to stress explicitly that his derivation was in no way dependent on any assumption concerning the ultimate nature of electricity – in particular, those that underlie Abraham's and Lorentz's theories. Clearly alluding to the point of view adopted in the paper he had published under Minkowski's name, Born wrote:

> *It must be emphasized that no use will be made here of atomistic hypotheses.* In fact, the atom or the electron, imagined as rigid bodies, can in no way be incorporated into the system of electrodynamics built on the principle of relativity, in which no analog is known of a rigid body in arbitrary accelerated motion. However, given the fact that all the basic expressions of Lorentz's theory of the electron seem to be independent of the hypotheses concerning the atomistic electron, the inertia of a continuously flowing charge can be likewise electromagnetically established in the sense suggested above. Naturally, this conception in no way contradicts those physical facts that indicate an extraordinarily strong, variable (almost atomistic) spatial distribution of matter and electricity.[65]

Born's second publication that year on the same topic is his better-known paper containing the definition of rigid bodies, submitted on June 13. Born asserted that his definition of rigidity would play a role in Maxwellian electrodynamics similar to that played by the classical rigid body in Newtonian mechanics. He was now ready to express opinions on fundamental issues openly, yet he preserved much of Minkowski's characteristic caution. His theory, he thought, accounted for the atomistic structure of electricity in a way that Abraham's theory did not. It thus corresponded to the "atomistic instinct" of so many experimentalists who found it very hard to support recent attempts to describe the movement of electricity as a fluid, unconstrained by any kinematic conditions, and affected only by the action of its own field.[66] On the other hand, in motivating this analysis Born did invoke concerns like those repeatedly stressed by Minkowski: to allow for a further clarification of the conceptual relationship between electrodynamics and the principle of relativity. This view, which is manifest in various places in Born's paper, is best encapsulated in the following passage:

> *The practical value of the new definition of rigidity must manifest itself in the dynamics of the electron.* The greater or lesser transparency of the results obtained by means of it will also be used, to a certain extent, for or against making the assumption of the principle of relativity universally valid, since experiments have not yet provided a definite proof of it and perhaps never will.[67]

[65] [1, pp. 572–573] (Italics in the original).

[66] [2, pp. 5–6].

[67] [2, p. 4] (Italics in the original).

1.6 Summary and Concluding Remarks

In this chapter I have argued that in order to understand the proper historical context of Minkowski's work on relativity one must consider it against the background of the ideas that animated Hilbert's program for the axiomatization of physics. In turn, Minkowski's work clarify the potential scope and possible applications of the principles of Hilbert's program, albeit in a direction that Hilbert did not cover – and could not have imagined – when he formulated the sixth problem of his Paris address in 1900 (a call for the axiomatizaion of physics) and even in teaching his 1905 course in Göttingen.

The assumption of universal validity of Lorentz covariance had been strongly suggested by experimental results obtained during the late nineteenth century, and its theoretical implications had been investigated from different perspectives in recent works, noticeably those of Lorentz, Poincaré and Einstein. Yet, in a spirit similar to that underlying Hilbert's program, Minkowski believed that the logical structure of the physical theories built on the principle of relativity had not been satisfactorily elucidated, and he set out to do so. He was interested in exploring the logical consequences of the principle and in proving that it does not contradict the existing edifice of the various disciplines of physics. The postulate of relativity should be taken as a further axiom appearing at the base of each and every physical theory, together with the particular axioms of that theory. Minkowski was able to prove for certain domains of physics that the ensuing theory indeed produced a consistent logical structure. For some other theories, such as gravitation, he was less successful, but he claimed to have showed at least that no contradiction had arisen by adding the principle, and that a consistent, Lorentz-covariant theory of gravitation could eventually be worked out in detail.

But the postulate of relativity was for Minkowski not simply an additional axiom, with perhaps a wider domain of validity in physics than others. It was an axiom of a different nature: a principle that should be valid for every conceivable physical theory, even those theories that were yet to be discovered or formulated. Minkowski compared the status of the postulate of relativity with that of the principle of conservation of energy, whose validity we assume even for yet unknown forms of energy. Interestingly, Einstein, too, had drawn a similar comparison at roughly the same time, between the principle of relativity and the second law of thermodynamics. Minkowski may have been aware of this, since it appeared in the *Annalen der Physik* as a reply to an earlier article of Paul Ehrenfest (1880–1933), who was then at Göttingen. But Einstein and Minkowski compared relativity and conservation of energy in different ways. Einstein spoke in his article of two "open" principles of physics, with a strong heuristic character. Unlike Minkowski and Hilbert, Einstein did not see the principle of relativity and the principle of energy conservation as parts of strictly deductive systems from which the particular laws of a given domain could be derived.[68] More generally, although Einstein introduced the principle of

[68] [8].

relativity together with the constancy of light at the beginning of his 1905 article as "postulates" of the theory (in some sense of the word), there are clear differences between Einstein's approach and Minkowski's axiomatic analysis of the postulate of relativity.[69] In fact, one of the main aims of Hilbert's program was to address situations like that raised by Einstein, which he saw as potentially problematic. As Hertz had pointed out in the introduction to his *Principles of Mechanics*, it has often been the case in the history of physics that, faced with conflict between an existing theory and new empirical findings, physicists have added new hypotheses that apparently resolve the disagreement but perhaps contradict some other consequences of the existing theory. Hilbert thought that an adequate axiomatic analysis of the principles of a given theory would help to clear away possible contradictions and superfluities created by the gradual introduction of new hypotheses into existing theories. This was essentially the same goal pursued by Minkowski: he sought to verify that the recent introduction of the principle of relativity into physics had not created such a problematic situation.

One of the central points that emerges from studying Minkowski's work within its proper context, and one which is strongly suggested by the proximity of Hilbert's program, is the idea that the place of the postulate of relativity in physics could be fully analyzed without assuming, and certainly without committing oneself to, any particular conception of the ultimate nature of physical phenomena. We may assume that, to the extent that he did take a definite position on the foundations of physics, he must have been close to some kind of mechanical reductionism, similar to that of Hilbert at the time. While there seems to be no direct evidence to answer this question, Minkowski's admiration for Hertz was consistently expressed and there is no evidence showing that he opposed him on this particular point.

The axiomatizing motivation behind Minkowski's work provides, then, a main perspective from which to understand the roots and the goals of his overall involvement with electrodynamics and relativity. This kind of motivation, however, appeared in combination with several other elements that informed his much more complex mathematical and physical background. The geometric element of this background, for instance, is one that has received much attention in the secondary literature, and must certainly be taken into account. Still, there are several reasons why one should be cautious in assessing its actual significance. For one, the very terms "geometry" and "geometrical" are much too comprehensive and sometimes imprecise. They need to be sharpened and placed in proper historical context if they are to explain in some sense Minkowski's motivations or the thrust of his articles on electrodynamics.[70] One should be able to describe, for instance, Minkowski's views on some of the basic, foundational questions of geometry and mathematics in general. We do not have much written evidence of this, besides the few

[69] On the other hand, Minkowski's axiomatic approach, and in particular his stress on universally valid principles in physics, strongly brings to mind Einstein's oft-quoted remarks on the differences between theories of principle and constructive theories. Cf. *CPAE* 2, *xxi–xxii*.

[70] A convincing analysis of the role of geometrical visualization in Minkowski's work in number theory appears in [51].

statements quoted at the beginning of this chapter that indicate a proximity to Hilbert's empiricist inclinations, and a stress on the significant, potential contributions of physical ideas to pure mathematics.

Elucidating the specific nature of Minkowski's conception of geometry becomes particularly important if we are to understand why, once he decided to undertake the axiomatic clarification of the role of the principle of relativity in physics, Minkowski came forward with a space-time geometry as an essential part of his analysis. Of primary interest in any discussion of this issue must be the connection between groups of transformations and geometry, which in "Space and Time", as was seen above, becomes a focal point of Minkowski's analysis. Klein was evidently very excited about this particular feature, and in a lecture of May 1910 he suggested, while referring to work done back in 1871, that he had in fact anticipated the approach behind Minkowski's study of the Lorentz group. The Minkowski space, he suggested, was just the four-dimensional version of a mathematical idea long familiar to himself, as well as to geometers like Sophus Lie (1842–1899) or Gaston Darboux (1842–1917).[71] On the other hand, when lecturing in 1917 on the history of mathematics in the nineteenth century, Klein remarked that among Minkowski's four papers he liked the first one most. Klein stressed the invariant-theoretic spirit of this paper as the faithful manifestation of Minkowski's way of thought.[72] Minkowski, for his part, did not mention Klein's ideas at all in his own articles, at least not explicitly. One may only wonder what would have been his reaction to Klein's assessments, had he lived to read them.[73] Although the connections suggested by Klein between his early geometrical work and the group-theoretical aspects of relativity in Minkowski's work may seem in retrospect clearly visible, there is no direct evidence that Minkowski was thinking literally in those terms when elaborating his own ideas on space and time.[74] Of course, the general idea that geometries can be characterized in terms of their groups of motions was by then widely accepted, and was certainly part and parcel of Hilbert's and Minkowski's most basic mathematical conceptions. An yet, one remarkable point that comes forward in my presentation is that, in the end, it was based on physical, rather than on purely mathematical considerations, that

71 Klein expressed these views in a meeting of the GMG, and they were published as [22].

72 [23, pp. 74–75], referring to [34]. Klein contrasted this paper with the *Grundgleichungen* in which – in order not to demand previous mathematical knowledge from his audience – Minkowski had adopted a more concise, but somewhat ad-hoc, matricial approach. The latter, Klein thought, was perhaps more technically accessible, but also less appropriate for expressing the essence of Minkowski's thoughts.

73 As already pointed out, the impact of some of some of Klein's work, particularly of the *Erlangen Programm* was somewhat overstated in many retrospective historical analyses, including those of Klein himself. See above §1.2, especially note 78.

74 For a discussion on the connection between Minkowski's space-time and the ideas associated with Klein's *Erlanger Programm* see [37, p.797]. Norton raises an important point when he claims that "the notion of spacetime was introduced into physics almost as a perfunctory by-product of the *Erlangen* program," but as indicated here, this formulation would seem to imply that program subsumed all the contemporary work on the relations between geometry and groups of transformations, an assumption that needs to be carefully qualified.

Minkowski's work helped consolidate the view that geometry is best understood in terms of the theory of groups of transformations.

The first to establish the explicit connection between the terminology and the ideas of group theory and the Lorentz covariance of the equations of electrodynamics was Poincaré, in his 1905 article. Remarkably, he had also been the first to use four-dimensional coordinates in connection with electrodynamics and the principle of relativity. Minkowski, on the other hand, was the first to combine all these elements into the new conception of the four-dimensional manifold of space-time, a conception that, however, emerged fully-fledged only in his 1908 Köln lecture and was absent from his earlier ones. What was the background against which Minkowski was led to take a step beyond the point that Poincaré had reached in his own work, and thus to introduce the idea of space-time as the underlying concept that embodies the new conception of physics? It is perhaps at this particular point that the specific impact of Einstein's work on Minkowski may have been decisive. One aspect of this work that Minkowski specifically singled out for its importance was Einstein's contribution to modifying the traditional concept of time; Minkowski proposed to do something similar for the concept of space, by replacing it with a four-dimensional geometry of space-time. A combination of this essential point taken from the original work of Einstein, together with the axiomatic perspective stemming from Hilbert's program may have provided the fundamental trigger leading to this innovation. Indeed, when explaining his motivation for studying kinematics with group theoretical tools, Minkowski asserted that the separation between kinematics and geometry had traditionally been assumed both in existing axiomatic analyses and in group-theoretical investigations. Hilbert had explicitly stressed in his axiomatization lectures that the axioms of kinematics would be obtained by coupling to the axioms of geometry, accounting for space, those required in order to account for the properties of time.

The subsequent development of the theory of relativity can hardly be told without referring to the enormous influence of Minkowski's contributions.[75] After an initial stage of indecision and critical responses, the space-time manifold as well as the four-vector language eventually became inseparable from the fundamental ideas introduced by Lorentz, Poincaré, and Einstein. Among the first to insist upon the importance of Minkowski's formulation were Max von Laue (1879–1960) and Sommerfeld. Sommerfeld, who had actually been among the earlier critics of Einstein's relativity, published two articles in 1910 that elaborated in a systematic fashion the ideas introduced by Minkowski and became the standard point reference for physicist over the coming years.[76] Laue published in 1911 the first introductory textbook on the special theory of relativity[77] that precisely because his use of Minkowski's formulation presented the theory in a level of clarity and

75 For an account of the immediate, varying responses among mathematicians and physicists, see [56], § 4.

76 [53].

77 [24].

sophistication that surpassed by far Einstein's original one. Einstein's initial reaction to Minkowski's work, was less enthusiastic, but he soon changed his attitude, and perhaps the influence of Laue and Sommerfeld may have been crucial in this respect.[78]

On the other hand, Minkowski's term "world-postulate", and the connotations implied by it, was never enthusiastically adopted,[79] and even less so was the kind of axiomatic analysis he performed for ensuring that the adoption of the world-postulate at the basis of any branch of physics would not lead to contradiction with the existing theories. And paramount among the existing theories for which the status of relativity remained unclear was gravitation. Physicists did not accord any special attention to Minkowski's more specific axiomatic treatment of the equations of electrodynamics for moving matter either. Hilbert, as usual, followed his own idiosyncratic path, and over the years following Minkowski's death he continued to insist in his lectures upon the need for an axiomatic treatment of physical theories, and to stress the importance of Minkowski's contribution in this regard. Eventually, when in 1915 Hilbert dedicated efforts to finding generally covariant field-equations of gravitation, he certainly saw himself as following in the footsteps of Minkowski's earlier work, not so much regarding the specific way the latter had attempted to formulate a Lorentz-covarianttheory of gravitation, but rather concerning the principles on which this attempt had been based. Still, the way from Minkowski's treatment of gravitation in 1908–1909 to Hilbert's treatment of the same matter in 1915 was anything but straightforward.

[78] In existing accounts, Einstein's alleged negative attitude towards Minkowski's work has sometimes been overemphasized. Thus, for instance, it has been repeatedly said that Einstein considered Minkowski's reformulation of his theory to be no more than "superfluous erudition" The source for this statement is [38, p. 151]. Pais, however, quotes no direct evidence, but rather attributes the claim to Valentin Bargmann (1908–1989), who reportedly heard it from Einstein. Bargmann, it must be emphasized, met Einstein for the first time in 1937. A second, oft-quoted statement in this direction attributes to Einstein the complaint that "since the mathematicians pounced on the relativity theory I no longer understand myself." Such a statement appears in [52, p. 46]. Einstein was also quoted as claiming that he could "hardly understand" Laue's book because of its strongly mathematical orientation, that followed very closely Minkowski's approach (cf. the introduction to the journal *Historical Studies of Physical Science* (*HSPS*) Vol. 7, *xxvii*, quoting [11, p. 206]. Einstein himself wrote in 1942 the preface of the German edition of Frank's book). Frank describes Einstein's claim (which is undocumented, in any case) as having been said "jokingly". The *HSPS* introduction already says "half-jokingly".

Written, relevant evidence that is available leads to different kind of emphases when describing Einstein's attitude in this regard. Thus for instance, Einstein and Laub [9, 10] do avoid the use of four-vectors and claim that Minkowski's mathematics is very difficult *for the reader*. Probably they did not favor Minkowski's formal approach at this stage, but they do not explicitly dismiss it either. In an unpublished article on the Special Theory of Relativity (STR) written in 1911 (*CPAE* 4, Doc.1), Einstein redid much of what appears in his collaboration with Jakob Laub (1882–1962), but now in four-dimensional notation. In fact, already in the summer of 1910, in a letter to Sommerfeld (*CPAE* 5, Doc. 211), Einstein explicitly expressed his increasing appreciation for the importance of such an approach. Cf. also a lecture of Jan. 16, 1991 – Einstein 1911. Whereas in January 1916, in a letter to Michele Besso (*CPAE* 8, Doc. 178), Einstein repeated that Minkowski's papers are "needlessly complicated", he could certainly have recommended a simpler and more elegant presentation in Laue's book.

As for Laue, Einstein consistently praised the high quality and clarity of his book. Cf. e.g., Einstein to Kleiner, April 3, 1912 (*CPAE* 5, Doc. 381). Moreover, in a manuscript written in 1912–1913, and published only recently (*CPAE* 4, Doc. 1, esp. §§3, 4), Einstein presents STR while following very closely the approaches of both Minkowski and Laue.

[79] Indeed, even Born, who was among the first to propagate Minkowski's formalism, did never come to use the term. Cf. [55, p. 293], footnote 67.

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