CHAPTER 10

From *Algebra* (1895) to *Moderne Algebra* (1930): Changing Conceptions of a Discipline–A Guided Tour Using the *Jahrbuch über die Fortschritte der Mathematik*

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Introduction

The discipline of algebra underwent significant changes between the last third of the nineteenth century and the first third of the twentieth. These changes comprised not only the addition of important new results, new concepts, and new techniques, but also fundamental shifts in the very way that the aims and scope of the discipline were conceived by its practitioners. During the nineteenth century, algebraic research had meant mainly research on the theory of polynomial equations and the theory of polynomial forms, including algebraic invariants. The ideas implied by Évariste Galois's works had become increasingly visible and central after their publication by Joseph Liouville in 1846. Together with important progress in the theory of fields of algebraic numbers, especially in the hands of Leopold Kronecker and Richard Dedekind, they gave rise to an increased interest in new concepts such as groups, fields, and modules.

A very popular textbook of algebra from the middle of the century was the *Cours d'algèbre supérieure* by Joseph Serret, which went through three editions in 1849, 1854, and 1866, respectively [Serret, 1849]. In these successive editions, this book gradually incorporated the techniques introduced by Galois, and in the third, it became the first university textbook to publish a full exposition of the theory. Still, it continued to formulate the main results of Galois theory in the traditional language of solvability dating back to the works of Lagrange and Abel at the beginning of the century. Thus, it did not even include a separate discussion of the concept of group. A second important, contemporary textbook was Camille Jordan's *Traité des substitutions et des équations algébriques* [Jordan, 1870], which already included a more elaborate presentation of the theory of groups, but which still treated this theory as subsidiary to the main tasks of algebra, and, above all, to the elucidation of solvability conditions for polynomial equations.

Towards the end of the century, Heinrich Weber published a three-volume textbook, *Lehrbuch der Algebra* [Weber, 1895] that incorporated an entire body of new ideas and techniques developed in the nineteenth century, thereby providing a full picture of what the body of algebraic knowledge looked like at the time. Concurrently, it implicitly embodied in the most elaborate and detailed way to date the disciplinary conception of algebra over the century. It laid down the main aims of this discipline, stressed the most relevant questions that practitioners had and should address, and presented the main available techniques available to do so successfully. In spite of the great amount of specific knowledge it added over books like Serret's or Jordan's, Weber's *Lehrbuch* did not embody an essentially different conception of the discipline from theirs;[1] algebra is seen as the discipline of polynomial equations and polynomial forms. Abstract concepts such as groups, in so far as they appear in the book, are subordinate to the main classical tasks of algebra. And, most importantly, all the results are based on the assumption of a thorough knowledge of the basic properties of the systems of rational and real numbers; these systems are conceived as conceptually prior to algebra. Whatever is said about polynomials or about factorization properties of algebraic numbers is based on what is known about the various systems of numbers.

The first two decades of the twentieth century were ripe with new algebraic ideas. Toward the end of the 1920s, one finds a growing number of works that can be identified with only recently consolidated theories, usually aimed at investigating the properties of abstractly defined mathematical entities now seen as the focus of interest in algebraic research: groups, fields, ideals, rings, and others. Like many other important textbooks, *Moderne Algebra*, written by the young Bartel L. van der Waerden, appeared in 1930 at a time when the need was felt for a comprehensive synthesis of what had been achieved since the publication of its predecessor, in this case Weber's *Lehrbuch*. It presented ideas that had been developed earlier by Emmy Noether and Emil Artin—whose courses van der Waerden had recently attended in Göttingen and Hamburg, respectively—and also by other algebraists, such as Ernst Steinitz, whose works van der Waerden had also studied under their guidance. Van der Waerden masterfully incorporated a great deal of the important innovations accumulated over the early decades of the twentieth century at the level of the body of algebraic knowledge. But the originality and importance of this book is best recognized by focusing on its totally new way of conceiving of the discipline. Van der Waerden presented systematically those mathematical branches then related to algebra, deriving all the relevant results from a single, unified perspective, and using similar concepts and methods for all those branches. This original perspective, which turned out to be enormously fruitful over the next decades of research—and not only in algebra, but in mathematics at large—is what I will call here the structural image of algebra.

The structural image of algebra as put forward in van der Waerden's textbook is based on the realization that a certain family of notions (that is, groups, ideals, rings, fields, etc.) are, in fact, individual instances of one and the same underlying idea, namely, the general idea of an algebraic structure, and that the aim of research in algebra is the full elucidation of those notions. None of these notions, to be sure, appeared as such for the first time in this book. Groups, as noted, had appeared in mainstream textbooks on algebra as early as 1866, in the third edition of

[1]Throughout this chapter, I will refer to the distinction between "body" and "images" of mathematical knowledge, on which I have elaborated in greater detail in [Corry, 2001; 2003]. Roughly stated, answers to questions directly related to the subject matter of any given discipline constitute the body of knowledge of that discipline, whereas claims and knowledge *about* that discipline constitute their images of knowledge. The images of knowledge help in discussing questions arising from the body of knowledge that are, in general, not part of, and cannot be settled within, the body of knowledge itself.

Serret's *Cours*. Ideals and fields, in turn, had been introduced in 1871 by Dedekind in his elaboration of Ernst Edward Kummer's factorization theory of algebraic numbers. But the unified treatment they were accorded in *Moderne Algebra*, the single methodological approach adopted to define and study each and all of them, and the compelling, new picture it provided of a variety of domains that were formerly seen as only vaguely related, all these implied a striking and original innovation.

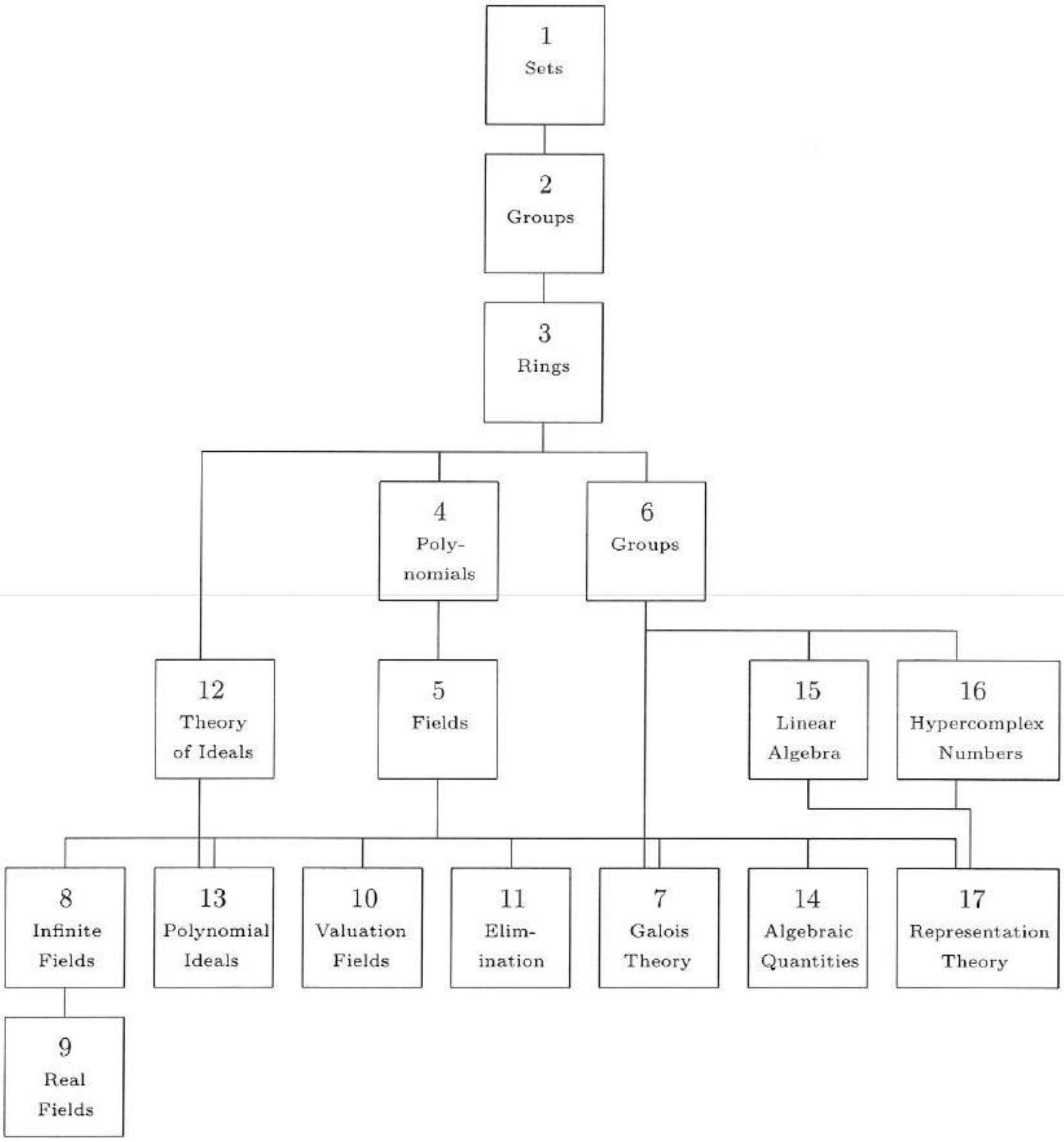
This is not the place to describe in detail the aims and contents of van der Waerden's book.[2] One fundamental innovation implied by his approach, however, that merits stressing here is the redefinition of the conceptual hierarchy underlying the discipline of algebra. Rational and real numbers no longer have conceptual priority over, say, polynomials. Rather, they are defined as particular cases of abstract algebraic constructs. Thus, for instance, van der Waerden introduced the concept of a field of fractions for integral domains in general, and then obtained the rational numbers as a particular case of this kind of construction, namely, as the field of quotients of the ring of integers. His definition of the system of real numbers in purely algebraic terms was based on the concept of a "real field," recently elaborated by Artin and Otto Schreier [Artin and Schreier, 1926], whose seminars van der Waerden had attended in Hamburg.

The task of finding the real and complex roots of an algebraic equation, which was the classical main core of algebra in the previous century, was relegated in van der Waerden's book for the first time to a subsidiary role. Three short sections in his chapter on Galois theory dealt with this specific application of the theory, and they assume no previous knowledge of the properties of real numbers. In this way, two central concepts of classical algebra (rational and real numbers) are presented here merely as final products of a series of successive algebraic constructs, the "structure" of which was gradually elucidated. On the other hand, additional, non-algebraic properties such as continuity and density were not considered at all by van der Waerden as part of his discussion of those systems.

Another important innovation implied by the book concerns the particular way in which the advantages of the axiomatic method are exploited in conjunction with all other components of the structural image of algebra, such as those mentioned above. Once one has realized that the basic notions of algebra (groups, rings, fields, etc.) are, in fact, different varieties of a same species ("varieties" and "species" understood here in a "biological," and not in a mathematical sense), namely, different kinds of algebraic structures, the abstract axiomatic formulation of the concepts becomes, in a natural way, the most appropriate one. The central disciplinary concern of algebra becomes, in this conception, the systematic study of those different varieties through a common approach. In fact, this fundamental recognition appears in *Moderne Algebra* not only implicitly, but rather explicitly and even didactically epitomized in the *Leitfaden* that appears in the introduction to the book, and that pictures the hierarchical, structural interrelation between the various concepts investigated in the book (see the next page).

Obviously, the new image of algebra presented by van der Waerden reflected the then-current state of development of the body of algebraic knowledge. However, the important point is that the image was *not a necessary* outcome of the body,

[2]For that, see [Corry, 2003, pp. 43-54].



but rather an independent development of intrinsic value. This becomes clear when we notice that parallel to van der Waerden's, several other textbooks on algebra were published which also contained most of the latest developments in the body of knowledge, but which essentially preserved the classical image of algebra. Examples of these are Leonard Eugene Dickson's *Modern Algebraic Theories* [Dickson, 1926], Helmut Hasse's *Höhere Algebra* [Hasse, 1926], and Otto Haupt's *Einführung in die Algebra* [1929]. But perhaps the most interesting example in this direction is provided by Robert Fricke's *Lehrbuch der Algebra*, published in 1924, with the revealing subtitle: "*Based on Heinrich Weber's Homonymous Book* [*Verfasst mit Benutzung vom Heinrich Webers gleichnamigem Buche*]" [Haupt, 1924]. These books were by no means of secondary importance. Dickson's, for instance, became after its publication the most advanced algebra text available in the United States, and it was not until 1941 that a new one, better adapted to recent developments in algebra and closer to the spirit of *Moderne Algebra*, was published there: *A Survey* *of Modern Algebra* by Garrett Birkhoff and Saunders Mac Lane [Birkhoff and Mac Lane, 1941].

Weber's *Lehrbuch* and van der Waerden's *Moderne Algebra*, then, embody in their respective presentations—thirty years apart—two very different images of the discipline they discuss. Faced with this fact, the question becomes how to account for the historical process that led from the former to the latter. The most immediate, and perhaps necessary, way to tackle this is to look at the most prominent works that, acting as milestones, progressively produced the main concepts, theorems, and techniques that came to stand at the center of algebraic research as it was practiced in the 1920s, and to explain how they helped shape the new images of algebra.[3] Parallel to this perspective, however, one may also look for additional hints that clarify how the practitioners of the discipline interpreted this progressive evolution and how their image of algebra changed accordingly. One way of illuminating this is to look at the leading, German review journal of the period, the *Jahrbuch über die Fortschritte der Mathematik* and, particularly, at the changing classificatory schemes adopted by the journal to account for the current situation at various, important crossroads of this story. As will be seen below, this perspective sheds interesting light on our understanding of the sometimes tortuous path from "Algebra" to "Modern Algebra."

The *Jahrbuch über die Fortschritte der Mathematik*

The *Jahrbuch über die Fortschritte der Mathematik* was established in 1868 by Carl Ohrtmann and Felix Müller, two Berlin Gymnasium teachers. It soon became the world's leading review journal of mathematics, to be eclipsed only after 1931 with the foundation in Germany of the *Zentralblatt für Mathematik und ihre Grenzgebiete* and later of the *Mathematical Reviews* in the United States beginning in 1940. The *Jahrbuch* published its last volume in 1945.[4] Other contemporary, but much less visible and influential review journals were the French *Bulletin des sciences mathématiques et astronomiques* founded in 1895 by Gaston Darboux, and the Dutch *Revue semestrielle des publications mathématiques* founded in 1897 [Siegmund-Schultze, 1993, pp. 14-20].

Although the advent of the *Jahrbuch* was enthusiastically welcomed by many in the German mathematical community, the degree of actual collaboration from that community's leading representatives was rather negligible. Some prominent German mathematicians did occasionally participate in writing reviews, but the truly outstanding names can hardly be found in the list.[5] The Berlin triumvirate of Kummer, Kronecker, and Karl Weierstrass, as well as their colleagues Carl Wilhelm Borchardt, Lazarus Fuchs, and Friedrich Schottky never contributed to the journal; Georg Frobenius and Hermann Amandus Schwarz did so only rarely. Felix Klein and David Hilbert, likewise, contributed very little, and they did so only before their great Göttingen years. Nor were any of the editors of the *Jahrbuch* truly first-rate mathematicians [Siegmund-Schultze, 1993, pp. 15-16 and 201-203].

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[3]This was the main topic of Part I of [Corry, 2003].

[4]For the history of the *Jahrbuch* and other mathematical review journals in the twentieth century, see [Siegmund-Schultze, 1993; 1994].

[5]Between 1900 and 1920, the quality of the reviewers considerably improved [Siegmund-Schultze, 1993, pp. 25-26].

The *Jahrbuch* was conceived as a yearly publication summarizing the relevant research activity of that period of time. It usually took several years, in some cases up to seven, before a given volume was completed and finally published. By the 1930s, this had become one of the journal's main drawbacks when compared to its competitors, the *Zentralblatt* and the *Mathematical Reviews* [Siegmund-Schultze, 1993, pp. 16-17].

It should come as no surprise that defining the classification schemes to be used for the mathematical works reviewed in the journal proved a challenging task from its inception. As Emil Lampe, co-editor of the *Jahrbuch* from 1885 until his death, explained "certainly there is no exhaustive classification of the mathematical disciplines, and although many groups can be easily demarcated on a coarse scale, it is extraordinarily hard to divide all the mathematical fields according to a precise uniform scheme. It is also quite easy for one to find fault in a division of too many categories with much too fine, puzzling principles of classification" [Lampe 1903, p. 5].[6] To state this in the terminology introduced above, adopting any given classification implies spelling out a certain image of mathematical knowledge that by nature is implicit, somewhat unstable, and only tacitly shared by some—but not necessarily all—members of the community. Moreover, specifying a classification scheme involves an attempt to freeze a conception that is essentially dynamic and that will certainly change as the body of knowledge changes. The editors of the *Jahrbuch* were clearly aware of this problem, and, over time, the tension was evident in their attempts, on the one hand, to preserve the existing schemes so as to make it easier for a reader to find articles and, on the other hand, to keep abreast of current developments in the body of knowledge and how they affect the images of mathematics.[7]

The use of the successive classificatory schemes of the *Jahrbuch* for algebra between 1900 and 1930 as an expression of the then-currently accepted images of the discipline must be set in the correct context by considering the background provided in the above account. For one thing, the schemes reflect the images of the editors of the journal at any given point in time, and there is no reason to believe that they were universally shared. For another, it is not absolutely clear how the schemes and the changes introduced in them were decided upon, whether by the editors alone, or by the editors in consultation with other mathematicians. Still, it would seem evident that the schemes attempted to express in the most coherent way possible what would appear to be an existing consensus in this regard, and thus they do express a certain common denominator that may have been shared by many contemporary mathematicians. More specifically, since the focus here is the gradual emergence of the structural conception of algebra and of the idea of an algebraic structure as a unifying principle across different algebraic subdisciplines, the absence of this idea as a leading classificatory component in the journal until the late 1920s may be taken, in my opinion, as a reliable criterion for its actual absence in the existing conceptions of the discipline, rather than only in the eyes of the editors of the *Jahrbuch*.

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[6]The English translation is quoted from [Despeaux, 2002, p. 298]. I thank Sloan Despeaux for providing me with a copy of her unpublished Ph.D. dissertation and for allowing me to quote from it.

[7]Compare [Despeaux, 2002, p. 299]. In her chapter 7, Despeaux charts the changes in the general classification schemes (not only in algebra) between 1868 and 1900.

Algebra by the Turn of the Century: The *Jahrbuch* in 1900

The image of algebra reflected in the classification scheme of the *Jahrbuch* at the beginning of the twentieth century is as close as it can be to that embodied in Weber's *Lehrbuch der Algebra*, as described above. Before presenting detailed evidence for this claim, however, it is illuminating to discuss briefly a preliminary instance, namely, an important and well-known article published by the same Weber in 1893 on "The General Foundations of Galois Theory" [Weber, 1893], and to see how it was classified in the relevant volume of the *Jahrbuch*.

In many respects, Weber's article represents the first truly modern published presentation of Galois theory, wherein the latter appears not just as an analysis of the problem of solvability, but rather as a more general examination of the interplay between specific groups and certain well-defined fields. In particular, Weber focused on establishing an isomorphism between the group of permutations of the roots of the equation and the group of automorphisms of the splitting field that leave the elements of the base field invariant. This approach leads in a natural way to adopting abstract formulations of the central concepts involved—group and field—while stressing the interplay between what we may in retrospect call their "structural" properties.

Weber, to be sure, was not the first to define groups abstractly, but this is, indeed, the first place where fields appear as an extension of the concept of group, obtained by adding a second operation to the already existing one. This permitted finite and infinite fields to be subsumed under a single, general definition, although, significantly, Weber did not consider the problem of the characteristic of the field [Corry, 2003, pp. 35-43].

The point that concerns us here is that the "structural" potentialities implied by the basic formulation of the theory in Weber's article were never fully exploited by him, and, in particular, his 1895 *Lehrbuch* contains no trace of them. Rather, Weber's book elaborated the classical, nineteenth-century approach to algebra in greater detail and comprehensiveness than ever before. This was, obviously, his true conception of the discipline. The incipient structural character present in his article strongly resonates because of subsequent developments, but it was of lesser importance for, and has less direct impact on, Weber himself and his contemporaries.

And, indeed, the *Jahrbuch* classifies this article in its 1893 volume according to classical standards, namely, as "Algebra: Equations (General Theory. Special Algebraic and Transcendental Equations)." In fact, all articles on Galois theory appeared in this section until much later, according to the classical conception that the theory is an auxiliary tool for dealing with the question of solvability of polynomial equations.

Consider now how algebraic works were classified in the *Jahrbuch* at the turn of the century, starting with volume 31 (1900).[8] This volume was published in 1902 and the editors were Emil Lampe and Georg Wallenberg. The two sections of particular interest here are Section II (Algebra) and Section III (Elementary and Higher Arithmetic). They are divided into subsections as follows:

[8]Very recently a web-based database comprising all the reviews published in the *Jahrbuch* was established as an "Electronic Research Archive for Mathematics" at http://www.emis.de/projects/JFM/JFM.html.

Section II: Algebra

Ch. 1: Equations: General Theory. Special Algebraic and Transcendental Equations

Ch. 2: Theory of Forms (Theory of Invariants)

Ch. 3: Substitutions and Group Theory. Determinants, Elimination and Symmetric Functions

Section III: Elementary and Higher Arithmetic

Ch. 1: Elementary Arithmetic

Ch. 2: Number Theory

A. General

B. Theory of Forms

Ch. 3: Continued Fractions (*Kettenbrüche*)

As in the *Lehrbuch*, this scheme acknowledges group theory specifically, but it is seen as closely related to substitutions and determinants, rather than, say, to fields, rings, or algebras. On the other hand, articles specifically dealing with "Galois Theory" all appear under the heading of "Special Algebraic and Transcendental Equations," since the theory is seen as simply one among several existing tools for dealing with the theory of equations, rather than as an autonomous topic or as one ancillary to the theory of groups.

Before the 1905 volume, there is only one noticeable change in the scheme adopted at the beginning of the century, and that appears in volume 34 (1903): the subsection on the theory of forms is divided there into (A) Theory of Algebraic Forms and (B) Differential Invariants. Representative of the articles included under the latter heading is a series of works by the Italian Ernesto Pascal [Pascal, 1903]. Pascal had published many similar articles prior to that date (for example, [Pascal, 1902]), and these were typically reviewed in the *Jahrbuch* in the section on differential and integral calculus under the sub-heading: "Partial Differential Equations." Thus, although the origins of problems in the theory of differential invariants lay in the domain of differential equations, the editors seem to have acknowledged in 1903 that the approach used in dealing with these problems was essentially similar to that used for algebraic invariants. Still, this is a rather minor change without overall implications for the disciplinary conception of algebra.

The classification scheme adopted in 1900 by the *Jahrbuch*, as already stressed, followed closely the image of algebraic knowledge put forward in Weber's *Lehrbuch*. Still, it was certainly possible at the time to follow that same image, while organizing algebraic knowledge slightly differently. Indeed, just at the turn of the century, the first chapters of Felix Klein's *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* began to appear in print [Gispert, 2001]. The articles composing the first volume of this collection, dealing with "Arithmetic and Algebra," were all published between 1898 and 1900, and it is illuminating to see how the then-current state of knowledge in algebra was distributed there among the various sections and articles. The volume was divided into three major sections on "Arithmetic," "Algebra," and "Number Theory," respectively, and these were further subdivided as follows:

A. Arithmetic

- Foundations of arithmetic (that is, elementary operations with numbers)
- Combinatorics
- Irrational numbers and convergence
- "Higher complex numbers" (that is, quaternions and hypercomplex systems)
- Set theory
- Finite groups

B. Algebra

- Rational functions of one variable
- Rational functions of several variables
- Algebraic forms. Arithmetic theory of algebraic magnitudes
- Algebraic invariants
- Separation and approximation of roots
- Rational functions of roots: symmetric functions
- Galois theory and applications
- Systems of equations
- Finite groups of linear substitutions

C. Number theory

- Elementary number theory
- Arithmetic theory of forms (that is, quadratic and bilinear forms, etc.)
- Analytic number theory
- Theory of algebraic number fields
- Cyclotomy fields
- Complex multiplication (that is, class field theory)

What better way to express implicitly the idea that algebra is above all the discipline dealing with the theory of polynomial equations and polynomial forms? The theory of algebraic number fields, for all its use of "algebraic" techniques and concepts, is seen here as part of a different (if neighboring) discipline, with stronger connections to analytic number theory than to, say, Galois theory. In turn, this latter theory is associated with the same mathematical family as the pursuit of analytical methods for approximating roots. There is no hint, of course, of the possibility of unifying under a separate, common heading works on groups, on fields or rings, or on associative algebras and hypercomplex systems. Not only do all of these concepts appear dispersed across the various subsections, but even the elementary theory of linear groups and the theory of linear groups appear under two different sections. It is also pertinent to notice that most of the chapters of the *Encyklopädie* were also individually reviewed as articles in the *Jahrbuch*, and they were often classified there under very different headings. Thus, for instance, the chapter on "Algebraic forms. Arithmetic theory of algebraic magnitudes" [Landsberg, 1899], that in the index of the *Encyklopädie* appears twice, once under "Algebra" and once under "Number Theory," is classified in the *Jahrbuch* as "Algebra: General Theory. Special Algebraic and Transcendental Equations."

Some Tentative Changes: 1905–1915

The first truly significant change in the classification scheme of the *Jahrbuch* for algebra after 1900 appeared in volume 36 (1905). The section on "Equations" became "Equations, Universal Algebra and Vector Analysis" and was further divided into (A) Equations and (B) Universal Algebra and Vector Analysis. While the first of these two subsections remained basically as before, the second includes topics that had never been covered under algebra, at least in any consistent way. Prominent in this subsection are works related to quaternions and vectors, among them, expository books such as Charles J. Joly's *A Manual of Quaternions* [Joly, 1905], and (in volume 37 (1906)) the second edition of Alexander MacFarlane's *Vector Analysis and Quaternions* [MacFarlane, 1906]. In earlier volumes, works of this kind had appeared variously under "Analytic Geometry: Textbooks, Coordinates" (for example, [Kelland and Tait, 1904]), "Differential and Integral Calculus: Determinate Equations" (for example, [Joly, 1903a]) or, in many cases, "Algebra: Equations" (for example, [Bucherer, 1903] and [Joly, 1903b]).

In order to understand more precisely the context and the difficulties faced by the editors with the choice of this, or any other scheme, however, it must be stressed that even with the availability of a new, special subsection for works dealing with quaternions and vectors, the 1905 volume still reflects a certain ambivalence concerning the most adequate way to classify them. Thus, for instance, while an article on "Hamilton's Quaternion Vector Analysis" [Knott, 1905] is classified in the new section on "Universal Algebra and Vector Analysis," one on "Quaternion Number-Systems" [Hawkes, 1905] is still classified as "Number Theory" and one on "Quaternion Products" [Stringham, 1905] under the more traditional heading of "Analytic Geometry: Coordinates." Only gradually were all articles on vectors and quaternions considered as belonging naturally to a single category, which is itself a subsection of "Algebra."

This change in the classificatory scheme of the *Jahrbuch* corresponds to parallel developments in the related body of knowledge. The last decade of the nineteenth century and the early years of the twentieth witnessed intense activity in the somewhat diverging approaches provided by quaternions and vectors to the same general set of topics. On the one hand, the "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics" published several issues of its *Bulletin* between 1900 and 1913 [Crowe, 1967, p. 218]. On the other hand, influential books that adopted the vectorial approach, especially through its application to physical theories, led to an eventual dominance of this approach and to a unification of all existing, related languages. This was the case with [Bucherer, 1903], [Gans, 1905], and [Marcolongo and Burali-Forti, 1907], as well as with influential articles in Klein's *Encyklopädie*, such as Abraham's article on the mechanics of deformable bodies [Abraham, 1901] or Lorentz's article on electron theory [Lorentz, 1904]. After 1905, the modern system of vector analysis had essentially been absorbed into the mainstream treatment of physical theories [Crowe, 1975, pp. 238–242], and, as the *Jahrbuch* makes clear, the more purely algebraic spirit implied by the approach became increasingly evident.

But besides quaternions and vectors, and perhaps more significantly for the purposes of the present account, among the articles covered under the new heading of "Universal Algebra" were the groundbreaking contributions of Joseph H. M. Wedderburn and Leonard E. Dickson to the study of finite algebras and hypercomplex systems, for example, [Wedderburn, 1905] and [Dickson, 1905a; 1905b].[9] Wedderburn's previous articles on related topics had been variously classified as "Algebra:

[9]Compare [Parshall, 1983].

Determinants" [Wedderburn, 1902] or "Algebra: Equations" [Wedderburn, 1903]. Dickson, on the other hand, had published an astonishingly large amount of work on group theory, especially linear groups, and some of his articles more closely related to algebras and hypercomplex systems had also appeared under the same heading of group theory (for example, [Dickson, 1903a; 1903b]), or sometimes simply under "Algebra: Equations" (for example, [Dickson, 1903c]). Another interesting example is provided by works of Issai Schur on the theory of matrices; in 1902 they were classified as "Algebra: Group Theory, Determinants," whereas in 1905 they already appeared under "Universal Algebra."

A different perspective on the meaning of the classification scheme is provided by works that would later be strongly identified with the modern, structural approach to algebra, for instance, Emanuel Lasker's important 1905 article on the factorization of polynomial ideals [Lasker, 1905].[10] Later generalized by the work of Macaulay (see below), Lasker's main results defined one of the central pillars, together with Dedekind's factorization theorems for algebraic integers, of Emmy Noether's abstract theory of rings. Lasker elaborated on ideas that had appeared in previous works of Hilbert on the theory of polynomial invariants. The article was reviewed in the chapter on the "Theory of Forms," but not in the section on "Algebra," rather in the section on "Elementary and Higher Arithmetic."

More significant is the case of Ernst Steinitz's seminal article on the abstract theory of fields [Steinitz, 1910]. The importance of this article for the rise of the structural approach in algebra can hardly be overstated. This is the first place where we find an analysis of the kind that can be dubbed "structural" with full justification of an algebraic entity that is defined in purely abstract terms. A purely abstract definition of field, as already mentioned above, had already been formulated in 1893 by Weber. However, as Steinitz pointed out in his article, beyond the definition itself, no truly abstract investigation of the theory implied by this concept had ever been put forward. Steinitz used for the first time truly set-theoretical considerations in a work of this kind (including a thoughtful application of the axiom of choice only at the right place, where it was truly necessary). He also based his whole analysis on a systematic discussion of the possible cases of the most basic kinds of fields (prime fields), and a study of the kinds of properties that are passed over from these basic fields to any extensions or subentities thereof.[11] One could even say that in writing his textbook of 1930, van der Waerden was actually extending to the whole of algebra the paradigm embraced by Steinitz in his study of fields.

It is thus all the more surprising to see where this article was reviewed in the 1910 volume of the *Jahrbuch*: it is not even classified in the section on algebra, but rather under "Function Theory: General." One wonders what criteria resulted in classifying the article this way. Or perhaps it was a simple technical mistake? Looking at the existing classification scheme of the journal at this time, however, it is unclear into what section it would have fit in a completely natural way. Such a subsection appeared only in 1916, following important developments in algebra for which Steinitz's article itself was a main driving force. This shift eventually resulted in a reclassification of a later reprint of Steinitz's article.

[10] Compare [Corry, 2003, pp. 214-219].

[11] See [Corry, 2003, pp. 192-196] for details.

Transactions of the American Mathematical Society: 1910

Additional insight may be gained at this point by looking at the few contemporary classificatory schemes used in venues other than the *Jahrbuch*. In the United States, for instance, there was no specialized review journal at the time; the *Mathematical Reviews* would only be established in 1940. Still, the index of the first ten volumes of the *Transactions of the American Mathematical Society* was published as an appendix to the 1909 volume of this leading American journal. There, the articles are classified according to topics and subtopics, and, in this sense, it is useful to compare it with the *Jahrbuch*'s scheme, in spite of its limited scope; it only covered works published in the journal itself.

The first section in the index refers to the "Logical Analysis of Mathematical Disciplines," which includes a relatively long list of works associated with the distinctly American tradition of "Postulational Analysis." This tradition derived from Hilbert's work on the foundations of geometry at the turn of the century, but took a twist of its own following the lead of Eliakim Hastings Moore. The idea was to take sets of postulates defining various mathematical entities pertaining to different disciplines and to analyze them using the tools developed by Hilbert in the *Grundlagen der Geometrie*. The focus was on the sets of postulates themselves, rather than on the specific disciplines to which they referred, and the expected results of works of this kind were the formulation of a minimal set of independent postulates underlying the various mathematical theories. This trend was very active in the United States until 1920, and involved contributions by the leading American mathematicians, such as E. H. Moore, Dickson, Robert L. Moore, and Edward Huntington. In Germany, on the other hand, it received scant attention.[12] It is therefore not surprising that a separate section in the index of the *Transactions* was specifically devoted to it. Nothing similar to this appears until 1916 in the *Jahrbuch*, that continued meanwhile to classify articles on postulational analysis according to the discipline whose postulates were analyzed, varying from "Differential and Integral Calculus" [Huntington, 1902], to "Philosophy" [Huntington, 1904], to "Groups" [Huntington, 1905] and [Dickson, 1905b], to "Universal Algebra" [Carstens, 1906], to "Mechanics" [Carmichael, 1912], to "Geometry" [Huntington, 1913].

Our main interest here, though, is the section on algebra in the index of the *Transactions*. This section is quite different from its counterpart in the *Jahrbuch*, reflecting once again more typically American conceptions of the discipline at the time. It is divided into four subsections, as follows:

- B1. Rational Functions. Theory of Equations. Determinants. Symmetric Functions
- B2. Algebraic Forms
- B3. Linear Associative Algebra. Hypercomplex Systems. Fields
- B4. Algebra of Logic

Especially worthy of attention are the two last subsections. Linear associative algebras and hypercomplex systems had traditionally been a focal point of interest for American mathematicians at least since the related works of Benjamin Peirce in 1881, with deeper roots in the British tradition originating with William R. Hamilton and having James Joseph Sylvester as the bridge between the two continents

[12] Compare [Corry, 2003, pp. 214-219] for details.

[Parshall, 1985, pp. 226-261]. It is thus no wonder to find these topics here in a special section. That section also offers, of course, a natural location for [Wedderburn, 1905], the important article on finite algebras, and also could have offered the natural classification of his seminal article on the structure of algebras [Wedderburn, 1907] had it been published in this journal.[13] In the *Jahrbuch*, these two articles of Wedderburn appeared under the new subsection on "Universal Algebra and Vector Analysis." Considering the algebra of logic as part of algebra occurred only much later in German classificatory schemes, although works specifically devoted to such topics had existed in German at least as early as 1890 with Ernst Schröder's *Vorlesungen über die Algebra der Logik* [Schröder, 1890]. This work had been reviewed in the 1890 volume of the *Jahrbuch* in the section on "History and Philosophy: Philosophy and Pedagogy." This classification was preserved for decades and included articles such as Huntington's postulational analysis of this discipline [Huntington, 1904], which in the index of the *Transactions* appeared both under "Logical Analysis of Mathematical Disciplines" and "Algebra: Algebra of Logic."

The theory of numbers received in the index of the *Transactions* a section separate from algebra. So did the theory of groups, in which intense activity traditionally existed in the United States. This latter section was further divided into three subsections dealing, respectively, with "Discrete Groups in General," "Linear Groups in Arbitrary or Special Fields," and "Continuous Groups." A last section related to algebraic topics, but grouped under a different heading in the index of the *Transactions*, concerned "Algebraic Geometry," which appears here not as part of "Algebra," but rather as part of "Geometry." Interestingly enough, articles belonging to what we would associate in retrospect with the discipline of "Algebraic Geometry," both in its German and Italian traditions, appear in the various volumes of the *Jahrbuch* under different chapters of the section on "Geometry," and more often than not under "Analytic Geometry."

It is worth mentioning, to conclude this section, that the index of the next ten volumes of the *Transactions*, published as an appendix to its 1919 volume, repeated essentially the 1909 scheme. Then, in the 1928 volume, where the next ten volumes were indexed, no classificatory scheme whatsoever was used, and the articles appeared simply in alphabetical order by author.

The *Jahrbuch* after 1916

The next important change in the classificatory scheme of the *Jahrbuch* appeared in its volume 46, published in 1923 under the editorship of Leon Lichtenstein and containing reviews of the mathematical activity of the years 1916–1917.[14] The classificatory scheme contained many innovations for all the topics covered, and this was particularly the case for the two topics, algebra and arithmetic, which appeared now unified as part of a single section. This section, "Algebra and Arithmetic," was further subdivided into the following nine chapters:

[13]Compare [Parshall, 1985, pp. 309–331].

[14]After Lampe's death in 1918, Arthur Korn edited a single issue of the journal, and then the job was immediately taken over by Lichtenstein. He was perhaps the most proficient of the mathematicians who held this position. Compare [Siegmund-Schultze, 1993, p. 202].

Ch. 1 Foundations of Arithmetic and Algebra. General
Ch. 2 Elementary Arithmetic and Algebra. Combinatorics
Ch. 3 Theory of Polynomials and Algebraic Equations
Ch. 4 Theory of Forms. Determinants. Theory of Invariants
Ch. 5 Group Theory. Abstract Theory of Fields and Modules
Ch. 6 Elementary Theory of Numbers. Additive Number Theory
Ch. 7 Arithmetic Theory of Forms
Ch. 8 Algebraic Number Theory. Analytic Number Theory
Ch. 9 Transcendental Numbers

The chapter on foundations of algebra and arithmetic represents the first reference, as a separate category, for articles in the tradition of "postulational analysis." As noted, this trend did not attract the same kind of attention in Germany that it did in the United States. Perhaps the first work to introduce it in the German literature was a relatively unknown textbook on algebra by Alfred Loewy [Loewy, 1915].[15] Loewy had published several related articles in the *Transactions* during the first decade of the century and, at the same time, had been the main reviewer for the *Jahrbuch* of articles connected with this trend. Among those who read Loewy's book and were influenced by him was his nephew Abraham Fraenkel, who in 1912 published an analysis of a set of postulates for defining Hensel's system of p-adic numbers. This article, in turn, eventually evolved into a series of works that mark the starting point of the abstract theory of rings. This may have been among the factors that triggered the addition of a separate section for articles of this kind. At any rate, typical of the works that were reviewed under this heading is a series of articles by the Berkeley mathematician, Benjamin Abram Bernstein. Whereas his article on the postulates of Boolean algebra [Bernstein, 1916] was now seen as dealing with the foundations of algebra, his earlier ones [Bernstein, 1911; 1913; 1914] had been reviewed in the sections on probability, elementary arithmetic, and philosophy, respectively.

The chapter on "Elementary Arithmetic and Algebra. Combinatorics" included works on elementary methods for calculating logarithms or roots, but also on combinatorics. In previous volumes, articles on combinatorics had appeared in the section on probabilities.

The most striking innovation implied by the new classification, and the one that matters most for the purposes of the present account, is the inclusion of a unified chapter for group theory and the "Abstract Theory of Fields and Modules." For the first time, the classification of algebraic topics adopted by the *Jahrbuch* implicitly indicated that domains of inquiry associated with the concepts of groups, fields, and modules, which had originated within separate contexts and which had been hitherto considered as basically different in their nature and aims, were best understood if seen as different manifestations of one and the same underlying general idea, the idea that we will later identify as that of algebraic structure. However, this is just an early indication of an ongoing process which at this point in time still showed many signs of continuity with previous conceptions. Thus, for instance, Galois theory was still included as a special subsection of chapter 3 on the "Theory of Polynomials and Algebraic Equations." The more modern conception of this theory attained its definitive form in the works of Artin in the late 1920s; Artin conceived

[15] See [Corry, 2003, pp. 196-201] for details.

it as the study of the interrelation among certain fields and their algebraic extensions, on the other hand, and their associated Galois groups with their subgroups, on the other hand. This became one of the prominent hallmarks of the structural conception of algebra as manifest in van der Waerden's book, but, clearly, here it was yet to be fully developed. Likewise, some works on the theory of matrices still appear in this volume of the *Jahrbuch* as part of the chapter on "Theory of Forms: Determinants" rather than as connected with rings or even vector spaces.

At any rate, one wonders what was the direct motivation behind the decision to introduce this new chapter on "Abstract Theory of Fields and Modules" at this point in time, since the absolute majority of the articles reviewed in it, at least in the 1916–1917 volume are, in fact, articles on group theory. Among the few that are not was one of Fraenkel's early expositions of the essentials of an abstract theory of rings [Fraenkel, 1916]. This was a natural continuation of Fraenkel's dissertation [Fraenkel, 1914], and, in fact, he had submitted it as his *Habilitationsschrift*. In the former work, Fraenkel had followed the paradigm of Steinitz's 1910 work on fields and applied it to a new mathematical domain, that of rings, similar to fields but also containing zero divisors. Thus, Fraenkel had proved that the investigation of the algebraic properties of any "separable ring" (which he defined according to the existing definition of separable fields) may be reduced to that of "simpler rings," namely, rings that in essence contain only one prime zero divisor. In 1916, Fraenkel attempted to extend to rings the full range of questions addressed previously by Steinitz for fields and, in particular, the question of how to characterize all possible, algebraic and transcendental extensions of a given ring thereof.[16] In spite of their close mutual connection, these two works of Fraenkel were reviewed in two volumes of the *Jahrbuch* that used different classification schemes, and thus, while the earlier one was classified as "Higher and Elementary Arithmetic: Number Theory," the second appeared under "Algebra: Group Theory. Abstract Theory of Fields and Modules (Systems of Hypercomplex Numbers)."

Some additional works reviewed in this same section on abstract algebraic theories help illustrate the import of the gradual change in the images of algebra introduced in 1916 in the *Jahrbuch*, for instance Macaulay's 1916 important tract on factorization of polynomials [Macaulay, 1916]. An earlier article in which Macaulay had dealt with similar issues, extending Lasker's results to what later became known as the Lasker-Macaulay Theorem, appeared in [Macaulay, 1913]. This article had been reviewed in the *Jahrbuch*, like Lasker's, in the category "Higher and Elementary Arithmetic: Number Theory." The 1921–1922 volume, published in 1925 and still using the same classification scheme, included in this section works of Schur and of Wolfgang Krull on the abstract theory of rings [Krull, 1922; Schur, 1922], and, of course, the path-breaking work of Emmy Noether on factorization theorems in this theory [Noether, 1921].[17] Also in 1921, Fraenkel published an additional work on the same topic [Fraenkel, 1921]. The following year two important works of Dickson on algebras, that could have previously been reviewed in the section on "Arithmetic," appeared now under the new general heading [Dickson, 1923a; 1923b].

All of these works are interesting for the purposes of the present article, and not only for the way in which they were classified in the *Jahrbuch*. In fact, their

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[16]See [Corry, 2003, pp. 201-213] for additional details.

[17]See [Corry, 2003, pp. 225-237].

importance lies in how they actively contributed to the increasing realization of the deep change in process in the conceptions of the various algebraic domains, as well as in the interrelations among them. The important results proved in these works—and the fruitful way in which they implemented abstract formulations of concepts and increasingly structural research methods—turned them into harbingers of the emerging, structural image of algebra. Thus, in volume 51 of 1925, the subsection of algebra under consideration here, "Group Theory: Abstract Theory of Fields and Modules. Group Theory. System of Hypercomplex Numbers," turned into simply "Group Theory. Abstract Algebra." In retrospect, one may wonder why such a classification appeared only in 1925. After all, abstract formulations of central concepts had been known for many decades in algebra. As this account has stressed, however, the real change in the conception of the discipline occurred only slowly, as manifested in the *Jahrbuch*'s classification. There, it was only in 1925 that a separate part of algebra was that part dealing with several theories, all of which were defined by similar, abstract methods and of all which covered, abstractly formulated, structural questions.

On the other hand, the 1925 classification of algebraic topics in the *Jahrbuch* contains besides the already mentioned chapter on groups and abstract algebra, an additional one on the "Theory of Ideals," including works by Hasse and Masaso Sono [Hasse, 1924; Sono, 1924]. Once articles on abstract rings and abstract fields were classified under a common heading, it might have been the case that works on the abstract theory of factorization in terms of ideals would also fall into that category. That this was not the case can be taken, in my view, as further evidence of how slowly the full import of the structural conception of algebra, as embodied in van der Waerden's book, was understood. The year 1930 is, of course, important in this story as it is the year of publication of van der Waerden's *Moderne Algebra.* The stage was adequately set for its appearance, in terms of the classification scheme in the *Jahrbuch*, and, indeed, the book was reviewed in the section on "Algebra and Arithmetic" in the appropriate chapter: "Group Theory. Abstract Algebra." The same section also provided a natural framework for an important expository article on the latest developments in algebra, published that year by Helmut Hasse under the title "Die moderne algebraische Methode [Modern Algebraic Methods]" [Hasse, 1930]. The article appeared in the *Jahresbericht der Deutscher Mathematiker-Vereingung*, following in a tradition of publication of comprehensive reviews of recent work in this journal.[18] And a further work of interest reviewed in the same section that year is a new edition of Steinitz's work on fields, published with comments by Reinhold Baer and the same Hasse [Steinitz, 1930]. The *Jahrbuch*'s review of this article reproduced the introduction written by Baer and Hasse, which emphasizes the importance of the new image of algebra and the seminal role played in 1910 by the article in bringing about its consolidation. The review stressed that the article represented the starting point of much important research in algebra, and that it had became not only a milestone in the development of this discipline but also "an excellent and absolutely essential introduction for anyone intent on devoting himself to the study of the new algebra."

Moderne Algebra, as noted, played a decisive role in extending these ideas to the whole of algebra, thus crystallizing and helping to spread among a broad audience

[18] See the editorial remarks in the *Jahresbericht der Deutscher Mathematiker-Vereinigung* 1 (1891), 12.

in the mathematical community the new image of the discipline at the center of which stood the idea of an algebraic structure. The classification schemes of the *Jahrbuch* show, however, that there was still room for uncertainty as to the details of this image, hence the slight changes in this scheme in subsequent years. In volume 61 of 1935, the chapter on "Abstract Algebra" is further subdivided into a first section on groups and a second on rings and fields, whereas in volume 63 of 1937, the second section includes lattices together with groups and fields. The idea of a lattice had made its initial appearance in the late nineteenth century in the works of Schröder and Dedekind, but an elaborate theory built around this concept in its abstract formulation developed only after 1935 with the works of Garrett Birkhoff [Birkhoff, 1935] and Oystein Øre [Øre, 1935; 1936], and certainly under the new structural spirit promoted by van der Waerden's book.[19] The first works of this kind were classified in the *Jahrbuch* together with rings and fields, but it gradually became apparent to the editors that it would be adequate to point out explicitly that the section reviewed dealt not only with works on rings and fields but also with works on lattices.

The 1939 volume of the *Jahrbuch*, published in 1941, presented a final version of the classification scheme for the section on arithmetic and algebra, which reflected, in fact, the view of the discipline that would remain standard for decades to come. It comprised the following sections:

Ch. 1 General and Combinatorics
Ch. 2 Linear Algebra. Theory of Invariants
Ch. 3 Polynomials and Algebraic Equations
Ch. 4 Group Theory
Ch. 5 Abstract Theory of Lattices, Rings and Fields
Ch. 6 Fields of Numbers and Functions
Ch. 7 Number Theory
Ch. 8 Diophantine Approximations and Transcendental Problems

Concluding Remarks

To conclude this overview of the development of the images of algebra between 1900 and 1930 as reflected in the classification schemes of the *Jahrbuch*, it is illuminating to describe briefly the schemes adopted by the two new mathematical review journals, the *Zentralblatt* and the *Mathematical Reviews* in their early volumes. These two journals soon superseded the *Jahrbuch* and, in many respects, they may be considered as portraying more modern images and conceptions of mathematics than their predecessor [Siegmund-Schultze, 1994]. Yet, at least inasmuch as algebra is concerned, the two new reviewing journals were slow fully to adopt the structural image of the discipline. Consider, for instance, the classification scheme of the first volume of the *Zentralblatt*, published in 1931 under the editorship of Otto Neugebauer. Works in algebra appear under the major heading of "Arithmetic, Algebra and Group Theory," which comprises the following subsections:

- Foundations of Arithmetic and Algebra
- Linear Algebra. Determinants. Bilinear and Quadratic Forms
- Algebraic Equations
- Group Theory

[19] See [Corry, 2003, pp. 259-268] for additional details.

- Algebraic Numbers, Field Theory, Galois Theory, Ideal Theory in Fields of Numbers and of Functions
- Abstract Theory of Rings. Hypercomplex Numbers
- Invariant Theory. Elimination of Polynomial Ideals
- Number Theory
- Analytic Number Theory. Dirichlet Series, Diophantine approximations

Thus, at a time when the *Jahrbuch* was already reflecting the new, structural image of algebra, as described above, in a much more consistent fashion, the *Zentralblatt* still hesitated on this issue. On the one hand, rings and hypercomplex systems were conceptually associated there, but on the other hand, they were still separated from both field theory and the abstract theory of ideals.

The first issue of the *Mathematical Reviews* appeared in January 1941. The main driving force behind the new American reviewing journal was, once again, Neugebauer, who had emigrated to America, establishing himself at Brown University. Issues of the journal appeared monthly and were collected as a yearly volume, of which an index was compiled and articles classified retrospectively. Thus, although some kind of division into topics is found in this index, it appears more as an *a posteriori* organization of what was published than as a preconceived idea of how algebra, as well as all other mathematical disciplines, should be organized and subdivided. The classification of algebraic issues was far from systematic and changed from issue to issue in this first volume. The second volume of the journal seems to have been compiled under a more systematic and preconceived classification scheme, and this scheme gives a clearer idea of how algebra was conceived in the initial stages of the *Reviews*. Thus, articles on algebra are classified according to three major categories, namely, "Abstract Algebra," "Linear Algebra," and "Equations." These are further subdivided as follows:

Abstract Algebra

- Lattices and Boolean Algebras
- Rings and Ideal Theory
- Fields and Algebras
- Galois Theory
- p-adic Theories
- Function Fields

Algebra: Equations

- Symmetric Functions
- Zeros
- Classical Galois Theory
- Systems of Equations, Elimination
- Special Equations

Linear Algebra

- Matrices, Determinants, General Theory
- Special Matrices, Determinants
- Hypercomplex Systems
- Linear Forms and Equations
- Quadratic and Bilinear Forms
- Forms of Higher Degree
- Characteristic Values, Elementary Divisiors

Beyond these basic categories, there was also a major section on the theory of groups, which is subdivided into the following: "Finite," "Abelian," "Abstract Representations," "Characters," "Continuous Topological," "Lie," "Crystallography," "Generalized." In addition, there were shorter sections on topics related to algebra but not seen as part of the hard core of the discipline such as "Algebraic Functions," "Algebraic Geometry," "Algebraic Invariants" (as part of a more general section on invariants), "Algebra of Logic," and "Algebraic Number Theory" (as part of a more general section on number theory).

Although very close to the conception of algebra embodied in van der Waerden's book, there are still interesting differences such as the separate status accorded to group theory in all its manifestations. The year of publication of the second volume of the *Mathematical Reviews*, 1941, was also the year of publication of Birkhoff and Mac Lane's *Survey*, a textbook whose widespread adoption in American universities helped bring about the adoption of the new image of algebra in the burgeoning community of algebraists in the United States [Birkhoff and Mac Lane, 1941].

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